Title: Fractional Stirling Numbers
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In my last post, I brought the following equation to light:

$$\begin{bmatrix} n \\ p \end{bmatrix} = \lim_{\alpha \to 0} \left(\frac{\partial_\alpha^p (\alpha)_n}{p!} \right) \tag{1}$$

It is worthwhile to consider if this Equation applies to fractional values of n and p as well as the positive integers. First, take $p = 0.5$, $n = 1$. For this, we break out Cauchy's fractional integral formula:

$$\left(J^\beta f \right)(x) = \frac{1}{\Gamma(\beta)} \int_0^x \mathrm{d}t \, (x-t)^{\beta - 1} f(t) \tag{2}$$

In this case, $\beta = 0.5$ and $f(\alpha) = \alpha$ (let $x \to \alpha$), then we take the integer derivative with respect to α. In these operations, we get:

$$\Gamma\left(\frac{3}{2}\right) \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} = \lim_{\alpha \to 0} \partial_\alpha^{\frac{1}{2}} \alpha = \lim_{\alpha \to 0} \partial_\alpha \left(J^{\frac{1}{2}} f \right)(\alpha) = \lim_{\alpha \to 0} \partial_\alpha \left(\frac{\alpha^{\frac{3}{2}}}{\Gamma\left(\frac{1}{2}\right)} \int_0^1 \mathrm{d}t \, (1-t)^{-\frac{1}{2}} t \right) = 0 \tag{3}$$

Examining the same case $n = 2$:

$$\Gamma\left(\frac{3}{2}\right) \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix} = \lim_{\alpha \to 0} \partial_\alpha \left(J^{\frac{1}{2}} \alpha (\alpha + 1) \right) = \lim_{\alpha \to 0} \partial_\alpha \left(\frac{\alpha^{\frac{3}{2}}}{\Gamma\left(\frac{1}{2}\right)} \int_0^1 \mathrm{d}t \, (1-t)^{-\frac{1}{2}} t (\alpha t + 1) \right) = 0 \tag{4}$$

And it seems that $p = 0.5$ ought to go to 0 for all positive integer n. However, now we will try for fractional n and $0 \leq p \leq 1$:

$$\Gamma(p+1) \begin{bmatrix} n \\ p \end{bmatrix} = \lim_{\alpha \to 0} \partial_\alpha^{\lceil p \rceil} \left(J^{1-(p-\lfloor p \rfloor)} (\alpha)_n \right) = \lim_{\alpha \to 0} \partial_\alpha^{\lceil p \rceil} \left(\frac{\int_0^1 \mathrm{d}t \, (1-t)^{-(p-\lfloor p \rfloor)} (\alpha t)_n}{\alpha^{p - \lfloor p \rfloor - 1} \Gamma(1 - (p - \lfloor p \rfloor))} \right) = 0 \tag{5}$$

Clearly this form runs into difficulty when trying to work with fractional values of p, as the limit for a derivative of $\alpha^x (\alpha)_n$ is evaluated for $0 < x < 1$, such that the simpler computational solution (where $x = 0$ in Equation 1) is not present. Furthermore, the added α^x term makes all limit computations go to 0 (except when x is integer), as given on the right side of Equation 5. From this observation, then, it seems reasonable to suggest that the derivative with respect to p is a derivative Dirac delta function around every integer value of p. To test this theory, we will investigate further by:

$$\begin{aligned} \lim_{p \to \lceil p \rceil^-} \partial_p \left(\Gamma(p+1) \begin{bmatrix} n \\ p \end{bmatrix} \right) &= \lim_{\alpha \to 0} \partial_\alpha^{\lceil p \rceil} \alpha \int_0^1 \mathrm{d}t \, (\alpha t)_n \lim_{p \to \lceil p \rceil^-} \partial_p \left(\frac{(\alpha - \alpha t)^{-(p - \lfloor p \rfloor)}}{\Gamma(1 - (p - \lfloor p \rfloor))} \right) \\ &= \lim_{\alpha \to 0} \partial_\alpha^{\lceil p \rceil} \alpha \int_0^1 \mathrm{d}t \, (\alpha t)_n \\ &\quad \lim_{p \to \lceil p \rceil^-} \left(\left(-\ln(\alpha - \alpha t) + \psi_0 (1 - (p - \lfloor p \rfloor)) \right) \frac{(\alpha - \alpha t)^{-(p - \lfloor p \rfloor)}}{\Gamma(1 - (p - \lfloor p \rfloor))} \right) \end{aligned} \tag{6}$$

where $p_0 - \lfloor p_0 \rfloor \to 1^-$ and $\lceil p_0 \rceil - p_0 \to 0^+$. Now let $\lfloor p \rfloor \to 0$, and $\lceil p \rceil \to 1$. Then:

$$\begin{aligned}\lim_{p\to 1^-} \partial_p \left(\Gamma(p+1)\begin{bmatrix} n \\ p \end{bmatrix}\right) &= \lim_{\alpha\to 0} \partial_\alpha \int_0^1 \mathrm{d}t\ (\alpha t)_n \lim_{p\to 1^-} \left(\frac{-\ln(\alpha) - \ln(1-t) + \psi_0(1-p)}{(1-t)\Gamma(1-p)}\right) \\ &= \lim_{\alpha\to 0} \partial_\alpha \int_0^1 \mathrm{d}t\ (\alpha t)_n \left(\lim_{p\to 1^-}\left(\frac{-\ln(\alpha) - \ln(1-t)}{(1-t)\Gamma(1-p)}\right) - \frac{1}{(1-t)}\right)\end{aligned} \tag{7}$$

which is going to infinity, on account of the following identity:

$$\int_0^1 \mathrm{d}t\, \frac{\ln^s(1-t)}{1-t} \lim_{\alpha\to 0} \partial_\alpha (\alpha t)_n \to \pm\infty$$

where s is a positive integer. A pair of identities give clear justification for the phenomena described previously:

$$\lim_{\gamma\to 0} \frac{1}{\Gamma(\gamma)} \int_0^x \mathrm{d}t\, \frac{f(t)}{(x-t)^{1-\gamma}} = f(x)$$
$$\lim_{\alpha\to 0} \alpha^q \ln^s(\alpha) = 0$$

for some arbitrary small, positive, non-zero q. While we have only showed this idea to hold for $p \to 1$ ($\lfloor p \rfloor = 0$ and $\lceil p \rceil = 1$), I conjecture that it holds for all p. These identities raise the philosophical question: is the idea of using limits for finding integer values of p from more sophisticated fractional calculus formulas justified? It seems like the answer and calculation changes depending on the order in which the limits are evaluated. It does seem clear that the moment that the α term fully disappears in front of the integral, that we acquire a different behaviour in the mathematics entirely. Thus, the idea of a discontinuity in the Stirling Numbers of the First Kind with continuously changing p seems supported if we can justify that Equation 1 can be used for continuous values of p. I am unsure how to further evaluate this hypothesis.

There is one more idea I would like to test about Equation 1 before finishing - that is, the implications of using Equation 1 in a summation to evaluate a Taylor Series. For example, we find:

$$\begin{aligned} (\alpha)_n &= \sum_{p=0}^{\infty} \frac{\left[\partial_\delta^p (\delta)_n\right]_{\delta\to t}}{p!} (\alpha - t)^p \\ \implies (\alpha)_n &= \sum_{p=0}^{\infty} \frac{\left[\partial_\delta^p (\delta)_n\right]_{\delta\to \alpha-1}}{p!} \\ \implies n! &= \sum_{p=0}^{\infty} \frac{\left[\partial_\delta^p (\delta)_n\right]_{\delta\to 0}}{p!} = \sum_{p=0}^{\infty} \begin{bmatrix} n \\ p \end{bmatrix} = \sum_{p=0}^{n} \begin{bmatrix} n \\ p \end{bmatrix} \\ (n+1)! &= \sum_{p=0}^{n} \frac{\left[\partial_\delta^p (\delta)_n\right]_{\delta\to 1}}{p!} \\ \implies n+1 &= \frac{\sum_{p=0}^{n} \frac{\left[\partial_\delta^p (\delta)_n\right]_{\delta\to 1}}{p!}}{\sum_{p=0}^{n} \begin{bmatrix} n \\ p \end{bmatrix}} \\ \implies \frac{(n+1)_k}{k!} &= \frac{\sum_{p=0}^{n} \frac{\left[\partial_\delta^p (\delta)_n\right]_{\delta\to k}}{p!}}{\sum_{p=0}^{n} \begin{bmatrix} n \\ p \end{bmatrix}} \end{aligned} \tag{8}$$

Now if we take the last line and put it on an alternating infinite series in k, we get:

$$\sum_{k=0}^{\infty} \frac{(-1)^k (n+1)_k}{k!} = \left(\frac{1}{2}\right)^{n+1} \tag{9}$$

The ultimate result thus stands as:

$$\sum_{k=0}^{\infty} \sum_{p=0}^{n} \frac{\left[\partial_\delta^p (\delta)_n\right]_{\delta \to k}}{p!} = \left(\frac{1}{2}\right)^{n+1} n! \tag{10}$$

This concludes this blog post.