Title: The Fourier PDE
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June 30, 2025

Consider the complex Fourier series:

$$g\left(x\right) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n \frac{x}{L}} \tag{1}$$

Two forms of the Fourier coefficient c_n for complex Fourier series is:

$$\begin{aligned} c_n &= \frac{1}{L}\int_0^L \mathrm{d}x' g\left(x'\right) e^{-2\pi i n \frac{x'}{L}} \\ c_n &= \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}x' g\left(x'\right) e^{-2\pi i n \frac{x'}{L}} \end{aligned} \tag{2}$$

Recall Leibniz integral rule for $h\left(t\right) = \int_{a(t)}^{b(t)} f\left(t,x\right) \mathrm{d}x$:

$$\frac{\partial h\left(t\right)}{\partial t} = f\left(t, b\left(t\right)\right) \frac{\mathrm{d}b\left(t\right)}{\mathrm{d}t} - f\left(t, a\left(t\right)\right) \frac{\mathrm{d}a\left(t\right)}{\mathrm{d}t} + \int_{a(t)}^{b(t)} \mathrm{d}x \frac{\partial f\left(t,x\right)}{\partial t} \tag{3}$$

Apply this formula to the two lines of Equation 2, differentiating in L:

$$\begin{aligned} L\frac{\partial c_n}{\partial L} + c_n &= g\left(L\right) - n\frac{\partial c_n}{\partial n} \\ L\frac{\partial c_n}{\partial L} + c_n &= \left(-1\right)^n \frac{g\left(\frac{L}{2}\right) + g\left(-\frac{L}{2}\right)}{2} - n\frac{\partial c_n}{\partial n} \end{aligned} \tag{4}$$

These formulas can be used to solve for $g\left(x\right)$ if c_n truly represents a Fourier Series. There are some noteworthy technical aspects of these equations:

- The derivative with respect to n assumes that n is a continuous variable, though we know in the application that it is an integer. Thus, there are $e^{-2\pi i n}$ or $e^{-\pi i n}$ terms in c_n that must be maintained when taking the derivative before we simplify them assuming that n is an integer.
- When solving for g, the n dependence must simplify out such that all that is left is L dependence. In fact, I suspect that the necessary and sufficient condition for a series defined by c_n to be considered a Fourier series is that Equation 4 is satisfied and g is solved for purely in terms of L and not n.

The PDEs defined in Equation 4 are solved by Equation 2 when $g(x)$ is known and c_n is unknown. However, the Equation 4 can also be very useful when c_n is known and $g(x)$ is unknown, since the function being integrated to yield c_n can be solved for. The major barrier from using this formulation to solve for a general sum depending only on the integer variable n is that the L dependence must be right to make the series a Fourier series (where the derivative with respect to L is properly taken to guarantee that $g(L)$ depends only on L) while also having the "hidden" n dependence properly accounted for (so that the derivative with respect to n is correct). Of course, for a general sum, we would use Equation 1 with $x = 0$ and $L = 1$, such that $g(0) = \sum_{n=-\infty}^{\infty} c_n$. Of course, if only the n is present in c_n, then we generally represent a series. It seems unclear to me whether for a given n dependence if there could be multiple L dependences that could give the same answer for the series defined in terms of the integer variable n once we set $L = 1$ and $x = 0$ after taking the derivatives (i.e. solving for $g(x = 0)$). One idea is to first consider odd functions, such that the second line of Equation 4 simplifies to solve for $c_n + n\frac{\partial c_n}{\partial n} + L\frac{\partial c_n}{\partial L} = 0$ (note that this c_n here is defined somewhat differently than in the first line of Equation 2, since the integral bounds are different between Line 1 and Line 2). We will delve deeper into this discussion in future posts - for now, we will conclude this post by verifying that the first line of Equation 4 is true for a general power term - $g(x) = x^s$, where s is a positive integer. First, define the falling Pochhammer Symbol:

$$(s)_m = (s)(s-1)\dots(s-m+1) \tag{5}$$

where $(s)_0 = 1$ always. The calculation is as follows:

$$\begin{aligned}
c_n &= -\frac{1}{2\pi i n} L^s e^{-2\pi i n} \sum_{m=0}^{s-1} \frac{(s)_m}{(2\pi i n)^m} - \frac{s!}{(2\pi i n)^{s+1}} L^s \left(e^{-2\pi i n} - 1\right) \\
L\frac{\partial c_n}{\partial L} &= -\frac{1}{2\pi i n} L^s e^{-2\pi i n} \sum_{m=0}^{s-1} \frac{(s)_m \, s}{(2\pi i n)^m} - \frac{s! \, s}{(2\pi i n)^{s+1}} L^s \left(e^{-2\pi i n} - 1\right) \\
n\frac{\partial c_n}{\partial n} &= L^s e^{-2\pi i n} \sum_{m=0}^{s-1} \frac{(s)_m}{(2\pi i n)^m} \left(\frac{m+1}{2\pi i n} + 1\right) + \frac{s! \, (s+1)}{(2\pi i n)^{s+1}} L^s \left(e^{-2\pi i n} - 1\right) + \frac{s!}{(2\pi i n)^s} L^s e^{-2\pi i n}
\end{aligned} \tag{6}$$

Having carried out the computation, we can now simplify and make n an integer, then combine into Equation 4:

$$\begin{aligned}
g(L) &= c_n + L\frac{\partial c_n}{\partial L} + n\frac{\partial c_n}{\partial n} \\
&= L^s \left(-\sum_{m=0}^{s-1} \frac{(s)_m}{(2\pi i n)^{m+1}} - \sum_{m=0}^{s-1} \frac{(s)_m \, s}{(2\pi i n)^{m+1}} + \sum_{m=0}^{s-1} \frac{(s)_m}{(2\pi i n)^m} \left(\frac{m+1}{2\pi i n} + 1\right) + \frac{s!}{(2\pi i n)^s}\right) \\
&= L^s \left(-\sum_{m=0}^{s-1} \frac{(s)_m \, (s-m)}{(2\pi i n)^{m+1}} + \sum_{m=0}^{s-1} \frac{(s)_m}{(2\pi i n)^m} + \frac{s!}{(2\pi i n)^s}\right) \\
&= L^s \left(-\sum_{m=0}^{s-1} \frac{(s)_{m+1}}{(2\pi i n)^{m+1}} + \sum_{m=0}^{s-1} \frac{(s)_m}{(2\pi i n)^m} + \frac{s!}{(2\pi i n)^s}\right) \\
&= L^s \left(-\frac{s!}{(2\pi i n)^s} + \frac{s!}{(2\pi i n)^s} + 1\right) \\
&= L^s
\end{aligned} \tag{7}$$

as expected. This verifies the topic formula for any power series. This concludes this blog post.