Title: Expanding on Even Function Fourier Series
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In the last post, we investigated the Fourier Series of the even function $g(x)$, and found that for $L = 2x$, the following equalities exist:

$$\begin{aligned} g(x) &= c_0(x) + 2\sum_{n=1}^{\infty} c_n(x)(-1)^n \\ g(x) &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (-1)^n n \frac{\partial c_n(x)}{\partial n} \\ c_n(x) &= \frac{1}{2x} \int_{-x}^{x} \mathrm{d}x'\, g(x')\, e^{-\pi i n \frac{x'}{x}} \\ c_n(x) &= (-1)^n g(x) - x\frac{\partial c_n}{\partial x}(x) - n\frac{\partial c_n}{\partial n}(x) \end{aligned} \tag{1}$$

First off, philosophically, not all $g(x)$ correspond to theoretically solvable $c_n(x)$. An example of such a $g(x)$ is the pathological function found by Paul du Bois-Reymond - it is an even, everywhere continuous, but nowhere differentiable function:

$$g(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

Thus, it is also expected that there would be no solution to the PDE in Equation 1 in the case that this $g(x)$ was substituted in. The following more general function that is not purely even or odd also has no Fourier Series definition:

$$g(x) = \sum_{n=1}^{\infty} 2^{-n} e^{-\frac{1}{(x-q_n)^2}}$$

where q_n is determined by a lacunary series over the rational numbers. This function is purely continuous and infinitely differentiable, but is nowhere analytic, and cannot be modelled over its full domain by a Fourier Series. In this case, we would also expect that substituting this $g(x)$ into the general form of the PDE (where we have $\frac{g(x)+g(-x)}{2}$ rather than simply $g(x)$) would produce no solution for $c_n(x)$. These observations are also consistent with our knowledge of 1st order PDEs, which when specifically posed, can have no theoretical solution. This is an interesting consistency in the Fourier logic / philosophy - we are finding that Fourier coefficients are sophisticated enough in their two variable formulation that they must be necessarily specified as a PDE. This is an interesting insight to keep in mind as we embark on future Fourier endeavours.

One computational goal that came to mind for me after completing last post is whether the limit definition of $g\left(x\right)$ (2nd line of Equation 1, which came from considering the other three results in Equation 1) can be expanded on by applying the three relations repeatedly to attain an infinite number of relations for $g\left(x\right)$ in terms of $c_n\left(x\right)$ and it's partial derivatives. Of course, we keep $g\left(x\right)$ even. To start, we consider:

$$\begin{aligned}
g\left(x\right) &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left(-1\right)^n n\frac{\partial c_n}{\partial n}\left(x\right)\\
&= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left(-1\right)^n n\,\partial_n\left(\left(-1\right)^n g\left(x\right)-n\,\partial_n c_n\left(x\right)-x\,\partial_x c_n\left(x\right)\right)\\
\implies g\left(x\right) &= \frac{2}{i\pi}\lim_{N\to\infty}\frac{1}{N^2}\sum_{n=1}^{N}\left(-1\right)^n n\,\partial_n\left(n\,\partial_n c_n\left(x\right)+x\,\partial_x c_n\left(x\right)\right)
\end{aligned}\tag{2}$$

In the last post, I found the following expressions for the partial derivatives of $c_n\left(x\right)$:

$$\begin{aligned}
x\,\partial_x c_n\left(x\right) &= x\tilde{c}_n\left(x\right)\\
n\,\partial_n c_n\left(x\right) &= \frac{-i\pi n}{x}\tilde{c}_n\left(x\right)\\
\text{where }\tilde{c}_n\left(x\right) &= \frac{1}{2x}\int_{-x}^{x}\mathrm{d}x'\,g\left(x'\right)x'\,e^{-\pi i n\frac{x'}{x}}
\end{aligned}\tag{3}$$

Note that because $g\left(x\right)$ is even, $c_n\left(x\right)$ is also even in n. However, since $\tilde{g}\left(x\right)=g\left(x\right)x$ is odd, $\tilde{c}_n\left(x\right)$ is odd in n. Substituting Equation 3 in Equation 2, we get:

$$\begin{aligned}
g\left(x\right) &= \frac{2}{i\pi}\lim_{N\to\infty}\frac{1}{N^2}\sum_{n=1}^{N}\left(-1\right)^n n\,\partial_n\left(\frac{-i\pi n}{x}+x\right)\tilde{c}_n\left(x\right)\\
&= \frac{2}{i\pi}\lim_{N\to\infty}\frac{1}{N^2}\sum_{n=1}^{N}\left(-1\right)^n\left(\frac{-i\pi n}{x}\tilde{c}_n\left(x\right)+\left(\frac{-i\pi n}{x}+x\right)n\,\partial_n\tilde{c}_n\left(x\right)\right)
\end{aligned}\tag{4}$$

The only term that survives is the n^2 term, and we get:

$$g\left(x\right)=\frac{2}{i\pi}\lim_{N\to\infty}\frac{1}{N^2}\sum_{n=1}^{N}\left(-1\right)^n\,n^2\,\partial_n^2 c_n\left(x\right)\tag{5}$$

It turns out that the $\frac{1}{N^2}$ iteration of the calculation does not work as expected, but we stick with this idea, as iterating again to $\frac{1}{N^3}$ gives a calculation that does work:

$$g\left(x\right)=3\left(\frac{1}{i\pi}\right)^2\lim_{N\to\infty}\frac{1}{N^3}\sum_{n=1}^{N}\left(-1\right)^n\,n^3\,\partial_n^3 c_n\left(x\right)\tag{6}$$

It turns out that the $\frac{1}{N^m}$ (seems) to work for all *odd* positive integer values of m. That is:

$$g\left(x\right)=\left(\frac{1}{i\pi}\right)^{m-1}\lim_{N\to\infty}\frac{m}{N^m}\sum_{n=1}^{N}\left(-1\right)^n\,n^m\,\partial_n^m c_n\left(x\right)\text{ for odd }m\tag{7}$$

which turns out as a consequence of the following algebra:

$$\lim_{N\to\infty}\frac{m}{N^m}\sum_{i=1}^{N} i^m = \lim_{N\to\infty}(Nm)\,\frac{1}{N}\sum_{i=1}^{N}\left(\frac{i}{N}\right)^m = \lim_{N\to\infty} Nm\int_0^1 \mathrm{d}x\, x^m = \lim_{N\to\infty}\frac{Nm}{m+1}$$

Thus, though I still intend to explore why the even m do not produce the correct calculation, the odd calculation is interesting enough to continue with for the time being. We continue as a sum over the terms of the series, where $m = 2q+1$ and letting $M \to N$:

$$g(x) = 4\lim_{N\to\infty}\sum_{q=0}^{N}\left(\frac{1}{\pi}\right)^{2q+1}\sum_{n=1}^{N}(-1)^n\,\left(\frac{n}{N}\right)^{2q+1}\,\partial_n^{2q+1}c_n(x)\text{ for odd } m \tag{8}$$

where:

$$\sum_{q=0}^{\infty}\frac{(-1)^q}{2q+1} = \frac{\pi}{4}$$

Summing mainly in q, and noting that:

$$\left(\frac{n}{N}\right)^p\partial_n^p c_n = \frac{\left(\frac{-i\pi n}{Nx}\right)^p}{2x}\int_{-x}^{x}\mathrm{d}x'\,g(x')\,(x')^p\,e^{-\pi i n\frac{x'}{x}}$$

meaning that (with $M \to N$):

$$\begin{aligned} g(x) = \lim_{N\to\infty}\sum_{n=1}^{N}(-1)^n\,\frac{2}{x} \\ \int_{-x}^{x}\mathrm{d}x'\,g(x')\,e^{-\pi i n\frac{x'}{x}}\sum_{q=0}^{2N+1}\left(\frac{nx'}{iNx}\right)^q\left(\frac{1-(-1)^q}{2}\right) \end{aligned} \tag{9}$$

$$\implies g(x) = \lim_{N\to\infty}\sum_{n=1}^{N}(-1)^{n-1}\,\frac{2i}{x}\int_{-x}^{x}\mathrm{d}x'\,g(x')\,e^{-\pi i n\frac{x'}{x}}\left(\frac{\frac{nx'}{Nx}+\left(\frac{nx'}{Nx}\right)^{2N+3}}{1+\left(\frac{nx'}{Nx}\right)^2}\right)$$

We can verify this relation for $x \to 0$:

$$\begin{aligned}\lim_{x\to0} g(x) &= \lim_{N\to\infty}\lim_{x\to0}\sum_{n=1}^{N}(-1)^{n-1}\,2i\int_{-1}^{1}\mathrm{d}x'\,g(xx')\,e^{-\pi i n x'}\frac{\frac{n}{N}x'+\left(\frac{n}{N}x'\right)^{2N+3}}{1+\left(\frac{n}{N}x'\right)^2} \\ \lim_{x\to0} g(x) &= \lim_{N\to\infty} g(0)\sum_{n=1}^{N}(-1)^{n-1}\,2i\int_{-1}^{1}\mathrm{d}x'\,e^{-\pi i n x'}\frac{\frac{n}{N}x'+\left(\frac{n}{N}x'\right)^{2N+3}}{1+\left(\frac{n}{N}x'\right)^2}\end{aligned} \tag{10}$$

As a first result, we can note that by substitution, we get:

$$\int_{-1}^{1}\mathrm{d}x'\,e^{-\pi i n x'}\frac{\frac{n}{N}x'+\left(\frac{n}{N}x'\right)^{2N+3}}{1+\left(\frac{n}{N}x'\right)^2} = -i\int_{-\frac{n}{N}}^{\frac{n}{N}}\mathrm{d}x'\,\sin\left(N\pi x'\right)\frac{x'+(x')^{2N+3}}{1+(x')^2}$$

However, for infinitesimal integration bounds ($n \ll N$), we get a non-zero answer. Overall, we've obtained:

$$\lim_{x\to0} g(x) = \lim_{N\to\infty} g(0)\sum_{n=1}^{N}(-1)^{n-1}\,\frac{2}{\frac{n}{N}}\int_{-\frac{n}{N}}^{\frac{n}{N}}\mathrm{d}x'\,\sin\left(N\pi x'\right)\frac{x'+(x')^{2N+3}}{1+(x')^2} \tag{11}$$

which I have numerically confirmed that:

$$\lim_{N\to\infty}\sum_{n=1}^{N}(-1)^{n-1}\ \frac{2}{\frac{n}{N}}\int_{-\frac{n}{N}}^{\frac{n}{N}}\mathrm{d}x'\ \sin\left(N\pi x'\right)\frac{x'+\left(x'\right)^{2N+3}}{1+\left(x'\right)^{2}}=1$$

Now we have done some small verification of this formula. We will restate our prized result:

$$g\left(x\right)=\lim_{N\to\infty}\sum_{n=1}^{N}(-1)^{n-1}\ 2\int_{-1}^{1}\mathrm{d}x'\,g\left(xx'\right)\sin\left(n\pi x'\right)\left(\frac{\frac{n}{N}x'+\left(\frac{n}{N}x'\right)^{2N+3}}{1+\left(\frac{n}{N}x'\right)^{2}}\right)\tag{12}$$

such that:

$$\lim_{N\to\infty}\sum_{n=1}^{N}(-1)^{n-1}\ 2\sin\left(n\pi x'\right)\left(\frac{\frac{n}{N}x'+\left(\frac{n}{N}x'\right)^{2N+3}}{1+\left(\frac{n}{N}x'\right)^{2}}\right)\sim\delta\left(x'-1\right)\tag{13}$$

where $\delta\left(x'-1\right)$ is a Dirac-Delta like function.

However, these sums can be done differently. For example, instead of multiplying Equation 7 by $\frac{(-1)^q}{2q+1}$, we can multiply by $\frac{(-1)^q\pi^{2q+1}}{(2q+1)!}$. Then, given the series solutions:

$$\sum_{q=0}^{\infty}\frac{\left(-1\right)^{q}\pi^{2q}}{(2q+1)!\left(2q+1\right)}=F_{22}\left(\frac{1}{2},1;\frac{3}{2},\frac{3}{2};-\frac{\pi^2}{2}\right)$$

where F_{22} is the hypergeometric function. The analog to Equation 8 that we obtain (for $g\left(x\right)$ non-singular for all finite x) is:

$$\begin{aligned}
g\left(x\right)&=\lim_{N\to\infty}\sum_{n=1}^{N}(-1)^{n}\,\frac{1}{2x}\int_{-x}^{x}\mathrm{d}x'\,g\left(x'\right)e^{-\pi in\frac{x'}{x}}\frac{\sum_{q=0}^{2N+1}\frac{\left(\frac{\pi nx'}{iNx}\right)^{q}}{q!}\left(\frac{1-(-1)^q}{2}\right)}{F_{22}\left(\frac{1}{2},1;\frac{3}{2},\frac{3}{2};-\frac{\pi^2}{2}\right)}\\
&=\lim_{N\to\infty}\sum_{n=1}^{N}(-1)^{n-1}\,\frac{1}{2x}\int_{-x}^{x}\mathrm{d}x'\,g\left(x'\right)\sin\left(\frac{n\pi x'}{x}\right)\frac{\sin\left(\frac{n\pi x'}{Nx}\right)}{F_{22}\left(\frac{1}{2},1;\frac{3}{2},\frac{3}{2};-\frac{\pi^2}{2}\right)}\\
&+\lim_{N\to\infty}\sum_{n=1}^{N}(-1)^{n-1}\,\frac{1}{2x}\int_{-x}^{x}\mathrm{d}x'\,g\left(x'\right)\sin\left(\frac{n\pi x'}{x}\right)\\
&\times\frac{(-1)^N\left(\frac{n\pi x'}{Nx}\right)^{2N+3}F_{12}\left(1;N+2,N+\frac{5}{2};-\left(\frac{n\pi x'}{2Nx}\right)^2\right)}{(2N+3)!F_{22}\left(\frac{1}{2},1;\frac{3}{2},\frac{3}{2};-\frac{\pi^2}{2}\right)}
\end{aligned}\tag{14}$$

$$\begin{aligned}
\implies 1&=\lim_{N\to\infty}\sum_{n=1}^{N}(-1)^{n-1}\,\frac{1}{2}\int_{-1}^{1}\mathrm{d}x'\ \sin\left(n\pi x'\right)\frac{\sin\left(\frac{n\pi x'}{N}\right)}{F_{22}\left(\frac{1}{2},1;\frac{3}{2},\frac{3}{2};-\frac{\pi^2}{2}\right)}\\
&+\lim_{N\to\infty}\sum_{n=1}^{N}(-1)^{n-1}\,\frac{1}{2}\int_{-1}^{1}\mathrm{d}x'\ \sin\left(n\pi x'\right)\\
&\times\frac{(-1)^N\left(\frac{n\pi x'}{N}\right)^{2N+3}F_{12}\left(1;N+2,N+\frac{5}{2};-\left(\frac{n\pi x'}{2N}\right)^2\right)}{(2N+3)!F_{22}\left(\frac{1}{2},1;\frac{3}{2},\frac{3}{2};-\frac{\pi^2}{2}\right)}
\end{aligned}$$

which I have confirmed numerically. Other series are possible - consider:

$$\sum_{q=0}^{\infty} \frac{(-1)^q}{(2q+1)^2} = G$$

where G is Catalan's constant, $G \approx 0.915965$. Then, the ensuing relations are:

$$\begin{aligned}
g\left(x\right) &= \frac{\pi}{G} \lim_{N\to\infty} \sum_{n=1}^{N} (-1)^n \frac{1}{2x} \int_{-x}^{x} \mathrm{d}x'\, g\left(x'\right) e^{-\pi i n \frac{x'}{x}} \sum_{q=0}^{N} \frac{\left(\frac{nx'}{iNx}\right)^{2q+1}}{2q+1} \\
&= \frac{\pi}{G} \lim_{N\to\infty} \sum_{n=1}^{N} (-1)^{n-1} \frac{1}{2x} \int_{-x}^{x} \mathrm{d}x'\, g\left(x'\right) \sin\left(\frac{n\pi x'}{x}\right) \arctan\left(\frac{nx'}{Nx}\right) \\
&+ \frac{\pi}{G} \lim_{N\to\infty} \sum_{n=1}^{N} (-1)^{n-1} \frac{1}{4x} \int_{-x}^{x} \mathrm{d}x'\, g\left(x'\right) \sin\left(\frac{n\pi x'}{x}\right) \\
&\times (-1)^N \left(\frac{nx'}{Nx}\right)^{2N+3} \Phi\left(-\left(\frac{nx'}{Nx}\right)^2, 1, N+\frac{3}{2}\right) \\
\implies 1 &= \frac{\pi}{G} \lim_{N\to\infty} \sum_{n=1}^{N} (-1)^{n-1} \frac{1}{2} \int_{-1}^{1} \mathrm{d}x'\, \sin\left(n\pi x'\right) \arctan\left(\frac{nx'}{N}\right) \\
&+ \frac{\pi}{G} \lim_{N\to\infty} \sum_{n=1}^{N} (-1)^{n-1} \frac{1}{4} \int_{-1}^{1} \mathrm{d}x'\, \sin\left(n\pi x'\right) \\
&\times (-1)^N \left(\frac{nx'}{N}\right)^{2N+3} \Phi\left(-\left(\frac{nx'}{N}\right)^2, 1, N+\frac{3}{2}\right)
\end{aligned} \tag{15}$$

where Φ is the Lerch Transcendental function. This leads to an interesting (though probably not very fashionable) expression for the Catalan constant:

$$\begin{aligned}
G = \pi &\lim_{N\to\infty} \sum_{n=1}^{N} (-1)^{n-1} \frac{1}{2} \int_{-1}^{1} \mathrm{d}x'\, \sin\left(n\pi x'\right) \arctan\left(\frac{nx'}{N}\right) \\
+ \pi &\lim_{N\to\infty} \sum_{n=1}^{N} (-1)^{n-1} \frac{1}{4} \int_{-1}^{1} \mathrm{d}x'\, \sin\left(n\pi x'\right) (-1)^N \left(\frac{nx'}{N}\right)^{2N+3} \Phi\left(-\left(\frac{nx'}{N}\right)^2, 1, N+\frac{3}{2}\right)
\end{aligned} \tag{16}$$

I have also confirmed this result numerically. This concludes this post.