Title: Computing a Class of Logarithmic Integrals
Author: Josh Myers

June 29, 2025

As mentioned in the Stirling Numbers post, a defining property of the Stirling Numbers of the First Kind is the following identity for integer k:

$$\ln^k(1-x) = \sum_{n=k}^{\infty} (-1)^k \frac{k!}{n!} \begin{bmatrix} n \\ k \end{bmatrix} x^n \tag{1}$$

Thus,

$$\ln^k(x) = \sum_{n=k}^{\infty} (-1)^k \frac{k!}{n!} \begin{bmatrix} n \\ k \end{bmatrix} (1-x)^n \tag{2}$$

Consider the Beta function, defined by:

$$\mathrm{B}(b+1, a+1) = \int_0^1 x^b (1-x)^a \mathrm{d}x = \frac{\Gamma(b+1)\,\Gamma(a+1)}{\Gamma(a+b+1)} \tag{3}$$

The convenient form for integer a and b is:

$$\mathrm{B}(b+1, a+1) = \frac{b!\,a!}{(a+b+1)!} \tag{4}$$

Now consider the general integral for integer a and b:

$$I = \int_0^1 \ln^a(x) \ln^b(1-x)\,\mathrm{d}x \tag{5}$$

Applying Equation 1 to both logarithmic terms, we get a double infinite series as:

$$\begin{aligned} \int_0^1 \ln^a(x) \ln^b(1-x)\,\mathrm{d}x &= (-1)^{a+b} a!\,b! \sum_{n=b}^{\infty} \sum_{m=a}^{\infty} \frac{\begin{bmatrix} n \\ b \end{bmatrix} \begin{bmatrix} m \\ a \end{bmatrix}}{n!\,m!} \int_0^1 x^n (1-x)^m \mathrm{d}x \\ &= (-1)^{a+b} a!\,b! \sum_{n=b}^{\infty} \begin{bmatrix} n \\ b \end{bmatrix} \sum_{m=a}^{\infty} \frac{\begin{bmatrix} m \\ a \end{bmatrix}}{(m+n+1)!} \end{aligned} \tag{6}$$

where the second line follows because of the application of Equation 4. Now we will evaluate the second sum in Equation 6. Recall the identity for the Stirling Numbers of the First Kind:

$$\begin{bmatrix} m \\ a \end{bmatrix} = \frac{1}{m} \left(\begin{bmatrix} m+1 \\ a \end{bmatrix} - \begin{bmatrix} m \\ a-1 \end{bmatrix} \right) \tag{7}$$

This last identity is applied as follows:

$$\sum_{m=a}^{\infty} \frac{\left[{m \atop a}\right]}{(m+n+1)!} = \left(\frac{1}{n+1}\right) \sum_{m=a}^{\infty} \left(\frac{\left[{m \atop a}\right]}{m!\,(m+1)\dots(m+n)} - \frac{\left[{m+1 \atop a}\right] - \left[{m \atop a-1}\right]}{m!\,(m+1)\dots(m+n+1)} \right) \tag{8}$$

The terms involving $\left[{m \atop a}\right]$ and $\left[{m+1 \atop a}\right]$ form a telescoping series that cancels all terms but the first, where $m = a$. We then arrive at:

$$\sum_{m=a}^{\infty} \frac{\left[{m \atop a}\right]}{(m+n+1)!} = \left(\frac{1}{n+1}\right) \left(\frac{1}{(a+n)!} + \sum_{m=a}^{\infty} \frac{\left[{m \atop a-1}\right]}{(m+n+1)!} \right) \tag{9}$$

Indexing the series down one term to $m = a - 1$, we find that the $\frac{1}{(a+n)!}$ cancels and we are left with:

$$\sum_{m=a}^{\infty} \frac{\left[{m \atop a}\right]}{(m+n+1)!} = \left(\frac{1}{n+1}\right) \sum_{m=a-1}^{\infty} \frac{\left[{m \atop a-1}\right]}{(m+n+1)!} \tag{10}$$

This is a very powerful general result, as it implies that the series for some integer power of a can be found by iteratively applying Equation 10 to attain a much simpler series in terms of $\left[{m \atop 0}\right] = \delta_{m0}$. Thus we find the general result:

$$\sum_{m=a}^{\infty} \frac{\left[{m \atop a}\right]}{(m+n+1)!} = \frac{1}{n!\,(n+1)^{a+1}} \tag{11}$$

With this general result, the series representation of the general logarithmic integral simplifies considerably:

$$\int_0^1 \ln^a(x) \ln^b(1-x)\, \mathrm{d}x = (-1)^{a+b} a!\, b! \sum_{n=b}^{\infty} \frac{\left[{n \atop b}\right]}{n!\,(n+1)^{a+1}} \tag{12}$$

Making reference to the post on the Stirling Numbers of the First Kind, all $\left[{n \atop b}\right]$ can be expressed in terms of a series of generalized harmonic numbers (call this H_b^{Σ}) multiplied by a factor $(n-1)!$. For example, $H_3^{\Sigma} = \frac{1}{2}\left(H_{n-1}^2 - H_{n-1}^{(2)}\right)$ This simplifies to:

$$\begin{aligned} \int_0^1 \ln^a(x) \ln^b(1-x)\, \mathrm{d}x &= (-1)^{a+b} a!\, b! \sum_{n=b}^{\infty} \frac{H_b^{\Sigma}}{n\,(n+1)^{a+1}} \\ &= (-1)^{a+b} a!\, b! \sum_{n=b}^{\infty} \left(\frac{H_b^{\Sigma}}{n} - \frac{H_b^{\Sigma}}{n+1} - \frac{H_b^{\Sigma}}{(n+1)^2} - \dots - \frac{H_b^{\Sigma}}{(n+1)^{a+1}} \right) \end{aligned} \tag{13}$$

As can be seen, now the problem of computing I (see Equation 5) has been reduced to a problem of computing sums of a series of harmonic numbers divided by various powers of $n \geq b$. This is a highly convenient and instructive way to compute this general integral for learners, and is thus valuable as a blog post. Computing the sums necessary to solve this problem requires special attention of the next blog post based on a paper by Zheng in 2007, who, though was not the discoverer of the method to attain the series solutions, gives a clearly written synopsis of the technique and the associated results. This concludes this blog post.