Title: Induction on a Continuous Variable
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Many of my recent posts have made use of induction as proof of various summation formulas on the integers. This post will be a brief examination of the philosophical idea of induction on integers with the aim of determining the analogous concept for a continuous variable. First off, proof by induction for a sum indexed by an integer variable involves three steps:

- **Hypothesis:** Develop an expression that you expect is true for the value of an unknown sum of at least one integer variable.
- **Base Case:** Show that the hypothesis is true for one value of the integer variable.
- **Induction:** Show that if the hypothesis is true for the nth variable value, then it must be true for the $(n+1)$th variable value.

An example of proof by induction method is my post on the binomial theorem and some of its elementary implications in combinatorics.

Now we will zoom in on the inductive step: the inductive step tends to be about showing that the hypothesis properly reflects the amount added to the sum taking $n+1$ terms given its truth for n terms. Thus, it verifies that the hypothesis reflects (for a variable number of terms) the new value of the sum derived from adding *one more term*, reflecting the *smallest unit of change* in the variable. This realization is an important philosophical step, because it entails that we can begin to consider the meaning and method behind inductive proofs on a continuous variable. A connection to be made is that *an integral* can be expressed as a Riemann sum, and can casually be thought of as an infinite sum over infinitesimals that when all summed give a (potentially) finite result. Hypothesize that $g(x)$ *could* obey the following equality in terms of a known $f(x)$:

$$g(x) = \int_0^x f(t)\,\mathrm{d}t \tag{1}$$

Consider that for some x_o, Equation 1 is true. Now, let us consider an infinitesimal unit of change, dx in the variable x:

$$\begin{aligned} \lim_{dx\to 0} g(x+dx) &= \int_0^{x+dx} f(t)\,\mathrm{d}t \\ &= \int_0^x f(t)\,\mathrm{d}t + \lim_{dx\to 0}\int_x^{x+dx} f(t)\,\mathrm{d}t \\ &= g(x) + \lim_{dx\to 0} f(x)\,dx \end{aligned} \tag{2}$$

where the second to third line is a result of the definition of the differential element. We will now see that if Equation 2 is to be true (and thus prove that a proposed $g(x)$ truly satisfies Equation 1), $g(x)$ must satisfy a version of the Fundamental Theorem of Calculus for the known $f(x)$:

$$\begin{aligned} f(x) &= \lim_{dx\to 0} \frac{g(x+dx) - g(x)}{dx} \\ \implies f(x) &= g'(x) \end{aligned} \tag{3}$$

This conclusion is alternatively reached through taking the Taylor series of $g(x+dx)$ about x and taking the limit as $dx \to 0$, eliminating all but the linear (zero and first derivative) terms. Such a method would then demonstrate that the inductive step (Equation 2) is only true if $f(x) = g'(x)$ when comparing the Taylor series with the third line in Equation 2. Thus, the Fundamental Theorem of Calculus in Equation 3 must be true (i.e. $f(x) = g'(x)$) in order for Equation 1 to be true by induction (Equation 2). This subtle proof is obvious even to most students, but leads to interesting insight into the fundamental results that scientists use regularly. First, we note that we must use the definition of the differential element (in integration) in order to move from line 2 to line 3 of Equation 2. Then, we must use the definition of the derivative to complete the induction step. Thus, we show that through using the definitions of the *differential element* and *differentiation*, we fully determine $g(x)$ given $f(x)$ through inductive reasoning. Furthermore, this makes sense in a conceptual perspective as well: the derivative describes how the integral curve's value changes as we move along x, which corresponds directly to knowledge about "what is added" (within a constant) relative to the previous value of x when the integral is computed for the smallest increment of x, dx. This concludes this (philosophical) post.