Title: The Fourier PDE
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Consider the complex Fourier Series coefficient:

$$c_n = \frac{1}{L}\int_0^L g\left(x'\right) e^{-2\pi i n \frac{x'}{L}} \mathrm{d}x' \tag{1}$$

In the last post, we derived the partial differential equation for c_n by taking the derivative of Equation 1 with respect to L:

$$g\left(L\right) = c_n + n\frac{\partial c_n}{\partial n} + L\frac{\partial c_n}{\partial L} \tag{2}$$

We verified this PDE for $g\left(x\right) = x^s$, where s is a non-negative integer. While this essentially verifies the PDE for any function expressible as a power series, I'd like to shed more light on the function c_n in terms of its two variables, n and L. In this post, we will show that though L seems like the more problematic variable when differentiating Equation 1, it is actually the far more simplistic and trivial variable to work with. First consider Taylor's theorem:

$$f\left(x\right) = \sum_{n=0}^{\infty} \frac{f^{(n)}\left(a\right)}{n!}\left(x-a\right)^n \tag{3}$$

where a is a reference point on the domain of $f\left(x\right)$. This equation assumes that $f\left(x\right)$ is infinitely differentiable - we will apply this restriction to $g\left(x\right)$. Now consider integrand of Equation 1 as a special case of Equation 3, where $a = 0$ (this is known as the Maclaurin Series):

$$\begin{aligned} c_n &= \frac{1}{L}\int_0^L \mathrm{d}x \sum_{n=0}^{\infty}\frac{x^n}{n!}\sum_{m=0}^{n}\binom{n}{m}\left(-\frac{2\pi i n}{L}\right)^{n-m} g^{(m)}\left(0\right) \\ &= \frac{1}{L}\int_0^L \mathrm{d}x \sum_{m=0}^{\infty}\frac{g^{(m)}\left(0\right)}{m!}x^m \sum_{n=m}^{\infty}\frac{\left(-\frac{2\pi i n x}{L}\right)^{n-m}}{\left(n-m\right)!} \\ &= \sum_{m=0}^{\infty}\frac{g^{(m)}\left(0\right)}{m!}\left(\frac{1}{L}\int_0^L \mathrm{d}x\, x^m e^{-2\pi i n\frac{x}{L}}\right) \\ &= \sum_{m=0}^{\infty}\left(\frac{L}{2\pi i n}\right)^m \frac{g^{(m)}\left(0\right)}{m!}\frac{\gamma\left(m+1, 2\pi i n\right)}{2\pi i n} \end{aligned} \tag{4}$$

where $\gamma\left(s,z\right) = \int_0^z x^{s-1}e^{-x}\mathrm{d}x$ is the lower incomplete gamma function. As can be seen by Equation 4, the variable that is sophisticated to differentiate is n, *not* L. To find series for $n\frac{\partial c_n}{\partial n}$, we use

Equation 4 for $h(x) = xg(x)$ such that $h^{(m)}(0) = m\left(g^{(m-1)}(0)\right)$ (since from Equation 1, the new integrand is simply $x'g(x')\,e^{-2\pi in\frac{x'}{L}}$ when differentiating in n). Then, we find:

$$\begin{aligned} n\frac{\partial c_n}{\partial n} &= -\left(\frac{2\pi in}{L}\right)\sum_{m=1}^{\infty}\left(\frac{L}{2\pi in}\right)^m \frac{g^{(m-1)}(0)}{(m-1)!}\frac{\gamma(m+1,2\pi in)}{2\pi in} \\ &= -\sum_{m=0}^{\infty}\left(\frac{L}{2\pi in}\right)^m \frac{g^{(m)}(0)}{m!}\frac{\gamma(m+2,2\pi in)}{2\pi in} \end{aligned} \tag{5}$$

The L derivative of Equation 4 is significantly easier:

$$\begin{aligned} L\frac{\partial c_n}{\partial L} &= \sum_{m=1}^{\infty}\left(\frac{L}{2\pi in}\right)^m \frac{g^{(m)}(0)}{(m-1)!}\frac{\gamma(m+1,2\pi in)}{2\pi in} \\ &= \sum_{m=0}^{\infty}\left(\frac{L}{2\pi in}\right)^{m+1} \frac{g^{(m+1)}(0)}{m!}\frac{\gamma(m+2,2\pi in)}{2\pi in} \end{aligned} \tag{6}$$

Meanwhile, the incomplete gamma function can be evaluated as follows:

$$\begin{aligned} \gamma(s+1,z) &= -\sum_{q=0}^{s-1} z^{s-q}\frac{s!}{(s-q)!}e^{-z} - s!\left(e^{-z}-1\right) \\ \implies \gamma(m+1,2\pi in) &= -m!\sum_{q=0}^{m-1}\frac{(2\pi in)^{m-q}}{(m-q)!} \end{aligned} \tag{7}$$

Now substituting Equation 7 into Equations 4, Equation 5, and Equation 6, we get:

$$\begin{aligned} c_n &= -\sum_{m=0}^{\infty}\sum_{q=0}^{m-1}\frac{L^m}{(2\pi in)^{q+1}}\frac{g^{(m)}(0)}{(m-q)!} \\ n\frac{\partial c_n}{\partial n} &= \sum_{m=0}^{\infty}\sum_{q=0}^{m}\frac{L^m}{(2\pi in)^{q}}\frac{g^{(m)}(0)}{(m+1-q)!}(m+1) \\ L\frac{\partial c_n}{\partial L} &= -\sum_{m=0}^{\infty}\sum_{q=0}^{m}\frac{L^{m+1}}{(2\pi in)^{q+1}}\frac{g^{(m+1)}(0)}{(m+1-q)!}(m+1) \end{aligned} \tag{8}$$

To evaluate the results of these calculations, we will verify that the series calculations are consistent with the PDE. First, consider the PDE under differentiation in terms of L, setting $L=0$ afterwards:

$$\left(g^{(k)}(L)\right)_{L=0} = (k+1)\left(\frac{\partial^k c_n}{\partial L^k}\right)_{L=0} + \left(\frac{\partial^k}{\partial L^k}\left(n\frac{\partial c_n}{\partial n}\right)\right)_{L=0} \tag{9}$$

Taking these derivatives of Equation 8, we get:

$$\begin{aligned} (k+1)\left(\frac{\partial^k c_n}{\partial L^k}\right)_{L=0} &= -(k+1)!\,g^{(k)}(0)\sum_{q=0}^{k-1}\frac{1}{(2\pi in)^{q+1}}\frac{1}{(k-q)!} \\ \left(\frac{\partial^k}{\partial L^k}\left(n\frac{\partial c_n}{\partial n}\right)\right)_{L=0} &= (k+1)!\,g^{(k)}(0)\sum_{q=0}^{k}\frac{1}{(2\pi in)^{q}}\frac{1}{(k+1-q)!} \end{aligned} \tag{10}$$

Combining these into Equation 9 and re-indexing the second line to be a sum from $q=-1\to k-1$, we get left side equals right side ($g^k(0) = g^k(0)$). These are the Taylor series coefficients for the general, infinitely differentiable $g(x)$. This concludes this blog post.