Title: Trig Function Geometry
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Let's analyze the plot made by the parameterized functions:

$$\begin{aligned} x(t) &= \cos\left(2\pi\left(\frac{t}{2\pi}\right)^{p}\right) \\ y(t) &= \sin\left(2\pi\left(\frac{t}{2\pi}\right)^{z}\right) \end{aligned} \tag{1}$$

Before investigating the problem and the plot, let's solve for $y(t)$ in terms of $x(t)$.

$$y(t) = \sin\left(2\pi\left(\frac{\arccos(x(t))}{2\pi}\right)^{\frac{z}{p}}\right) \tag{2}$$

Thus, given that $-1 \leq x(t) \leq 1$, the value of $y(t)$ depends only on the fraction $\frac{z}{p}$. Plotting this function with $\frac{z}{p} = 2.8343$ (will explain this decision soon), we find: I will explain quickly how this

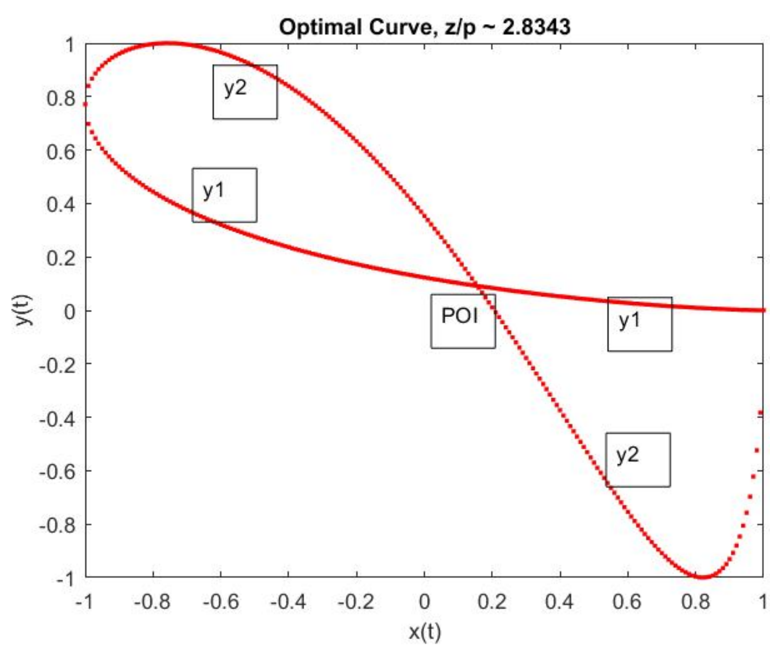

Figure 1: Geometry we'd like to analyze.

curve is produced. First, define:

$$y_1(t) = \sin\left(2\pi\left(\frac{\arccos(x(t))}{2\pi}\right)^{\frac{z}{p}}\right)$$
$$y_2(t) = \sin\left(2\pi\left(\frac{2\pi - \arccos(x(t))}{2\pi}\right)^{\frac{z}{p}}\right) \tag{3}$$

where y_2 and y_1 differ because the inverse cosine ranges only from 0 to π. Now the question can be posed: for what value of $\frac{z}{p}$ is the area contained between y_2 and y_1 equal on each side of the POI. Put succinctly, what value of $\frac{z}{p} = r_0$ is the following integral 0?

$$I_1(r_0) = \int_{-1}^{1} \mathrm{d}x(t)\left(\sin\left(2\pi\left(\frac{2\pi - \arccos(x(t))}{2\pi}\right)^{r_0}\right) - \sin\left(2\pi\left(\frac{\arccos(x(t))}{2\pi}\right)^{r_0}\right)\right) = 0 \tag{4}$$

It turns out that through numerics, $r_0 = 2.8343$, just as we plotted above! So the plot above shows approximately the curve where the two areas (one on each side of the POI) have equal area. Now that we've come this far, we can have more fun with similar problems related to this geometry. For example, what is the value of the following integral?

$$I_2(r_0) = \int_{-1}^{1} \mathrm{d}x(t)\frac{1}{\sin\left(2\pi\left(\frac{2\pi-\arccos(x(t))}{2\pi}\right)^{r_0}\right) - \sin\left(2\pi\left(\frac{\arccos(x(t))}{2\pi}\right)^{r_0}\right)} = \int_{-1}^{1} \mathrm{d}x(t)\, q(x(t)) \tag{5}$$

This integral can only be understood in terms of the Cauchy Principal Value. Evaluated numerically, we carry out the following problem:

$$I_2(r_0) = \lim_{\Delta x \to 0}\left(\int_{-1+\Delta x}^{x_0-\Delta x} \mathrm{d}x(t)\, q(x(t)) + \int_{x_0+\Delta x}^{1-\Delta x} \mathrm{d}x(t)\, q(x(t))\right) \tag{6}$$

Numerically, this value is found to converge for Δx as small as 10^{-10}, where x_0 (the POI) is found using a root finding algorithm on Equation 4. Knowing this, approximately, $I_2(r_0) = 1.2541$, found computationally. Now, what value of $\frac{z}{p}$ maximizes the value of I_2? This is also found numerically, by applying Equation 6 with a linear search in the value of the $\frac{z}{p}$ as a single parameter optimization problem. It turns out that $I_2(r)$ increases steadily as $\frac{z}{p}$ increases (see Figure 2 on the next page). There is one more interesting question I'd like to demonstrate, that is the following - what values of $\frac{z}{p}$ gives local maxima for the total area contained within the curve represented by Equation 3:

$$I_3(r) = \int_{-1}^{1} \mathrm{d}x(t) \mid y_2(x(t)) - y_1(x(t)) \mid \tag{7}$$

Figure 3 corresponds to the value of $I_3(r) = A$, where $r = \frac{z}{p}$. The y vs. x curves for the maxima (and the unit circle, $\frac{z}{p} = 1$) are plotted in Figure 4. The y vs. x curves for the minima are plotted in Figure 5.

Figure 4 and Figure 5 are interesting because they show snapshots of the evolution of the parameterized curve in Equation 3 for $\frac{z}{p} = r$, $r > 0$. Based on my search of the one-dimensional function space, beyond the maximum point with the highest $\frac{z}{p}$ ($\frac{z}{p} \approx 4.556$), the function seems to decrease monotonically with increasing $\frac{z}{p}$, approaching 0. It is interesting that there is a value of $\frac{z}{p} \approx 0.98$ that has slightly greater area contained in the curve than the $\frac{z}{p} = 1$, the unit circle. This concludes this blog post.

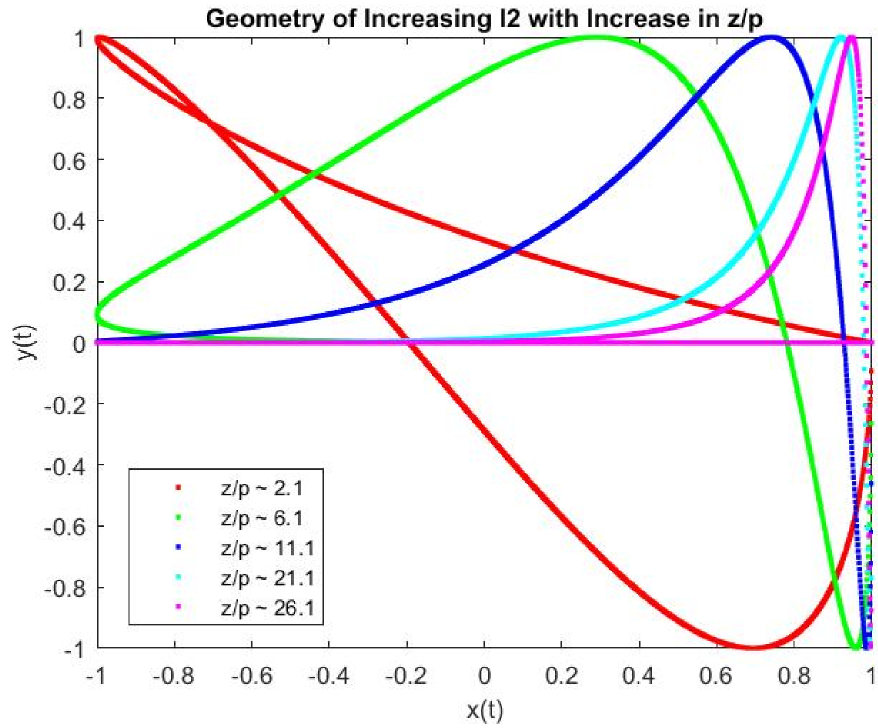

Figure 2: The result of scaling $\frac{z}{p}$ on the parameterized curve represented by Equation 3.

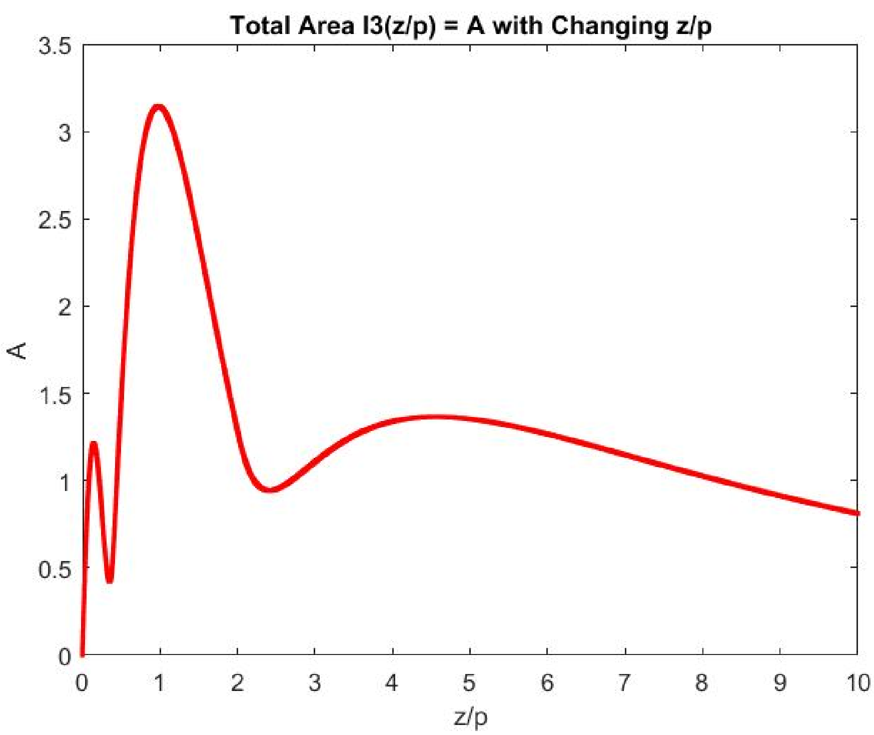

Figure 3: The variation of $I_3\,(r)$ as one varies $\frac{z}{p}$ from 0 to 10.

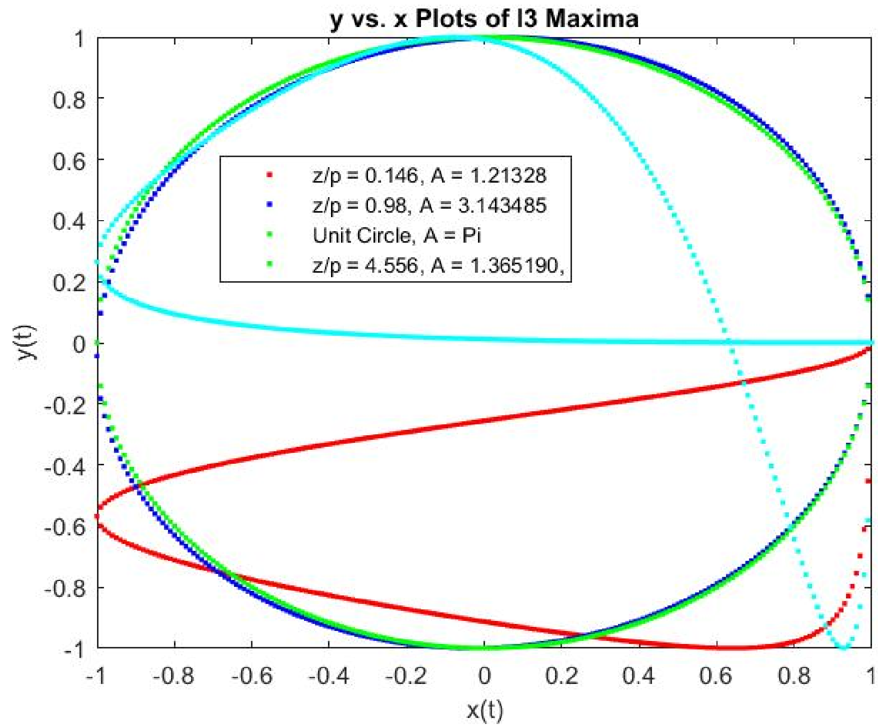

Figure 4: Equation 3 plots of the local maxima points of $I_3(r)$ from Figure 3.

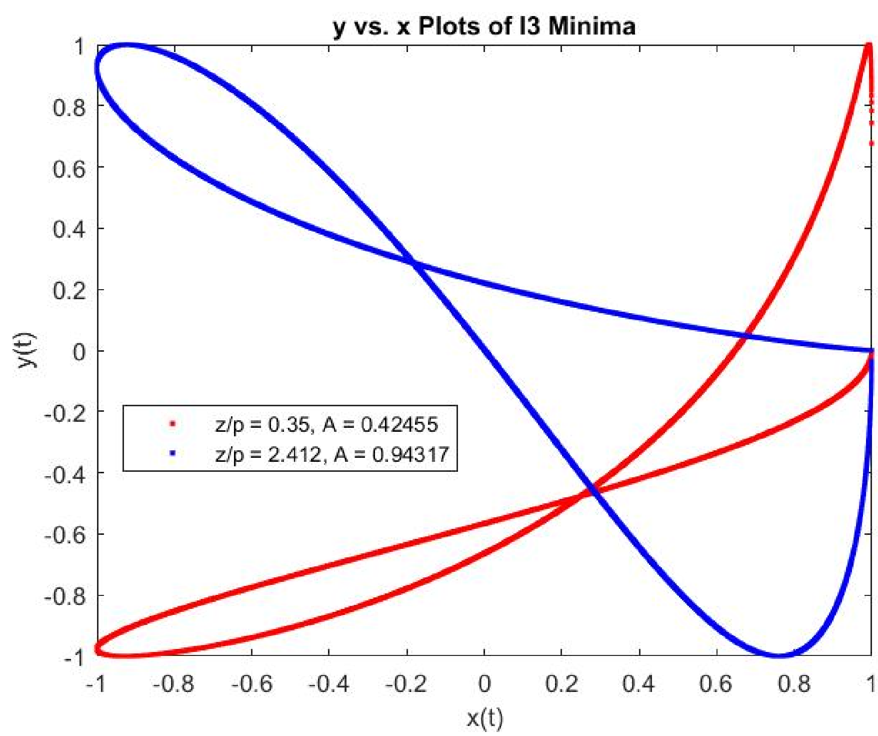

Figure 5: Equation 3 plots of the local minima points of $I_3(r)$ from Figure 3.