Title: Unsigned Stirling Numbers

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An identity for the logarithmic power series has been found as:

$$\ln^k(1-x) = \sum_{n=k}^{\infty} (-1)^k \frac{k!}{n!} \begin{bmatrix} n \\ k \end{bmatrix} x^n \tag{1}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ is a Stirling number of the first kind. Now differentiate Equation 1:

$$\begin{aligned} \frac{\ln^{k-1}(1-x)}{1-x} &= \sum_{n=k}^{\infty} (-1)^{k-1} \frac{(k-1)!}{(n-1)!} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-1} \\ \implies \sum_{m=k}^{\infty} (-1)^{k-1} \frac{(k-1)!}{(m-1)!} \begin{bmatrix} m-1 \\ k-1 \end{bmatrix} x^{m-1} \sum_{r=0}^{\infty} x^r &= \sum_{n=k}^{\infty} (-1)^{k-1} \frac{(k-1)!}{(n-1)!} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-1} \\ \implies \sum_{m=k}^{\infty} \sum_{r=0}^{\infty} \frac{1}{(m-1)!} \begin{bmatrix} m-1 \\ k-1 \end{bmatrix} x^{m+r} - \sum_{n=k}^{\infty} \frac{1}{(n-1)!} \begin{bmatrix} n \\ k \end{bmatrix} x^n &= 0 \end{aligned} \tag{2}$$

The coefficients must be equal for each power (for all combinations in which $m+r=n$, leaving a series identity for the Stirling numbers of the first kind:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{m=k}^{n} \frac{(n-1)!}{(m-1)!} \begin{bmatrix} m-1 \\ k-1 \end{bmatrix} \tag{3}$$

This identity is given on the Wikipedia page for the Stirling Numbers of the First Kind. This identity is particularly useful because it relates the nth Stirling Numbers of order k in terms of only the Stirling Numbers of lower order $(k-1)$ and lower $(m-1)$. This identity is useful also because it expresses the generation of the Stirling Numbers in terms of Stirling Numbers divided by the factorial:

$$\frac{\begin{bmatrix} n \\ k \end{bmatrix}}{n!} = \frac{1}{n} \sum_{m=k}^{n} \frac{\begin{bmatrix} m-1 \\ k-1 \end{bmatrix}}{(m-1)!} \tag{4}$$

where $\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} = n!$, and thus $\begin{bmatrix} n \\ k \end{bmatrix} \leq n!$, since $\begin{bmatrix} n \\ k \end{bmatrix} \geq 1$ for all n and k. Thus, Equation 4 can be useful to compute the Stirling Numbers of the first kind in terms in terms of $0 < \frac{\begin{bmatrix} n \\ k \end{bmatrix}}{n!} < 1$. In this way, there are strict bounds on the values generated by Equation 4, which is useful when evaluating the Stirling Numbers of the First Kind for Equation 1, which includes the factorial divisor. As well, as we will see next, Equation 3 is quite useful for generating explicit expressions for the Stirling

Numbers of the First Kind, since the kth solution can be found in terms of the previous $(k-1)$th solutions. We start with $k=0$ solution. When $k=0$, the only non-zero term is for $n=0$. Thus,

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = \delta_{n0} \tag{5}$$

Now, Equation 3 can be used to solve for $k=1$ case in terms of the $k=0$ case:

$$\begin{aligned} \begin{bmatrix} n \\ 1 \end{bmatrix} &= \sum_{m=1}^{n} \frac{(n-1)!}{(m-1)!} \delta_{(m-1)0} \\ &= (n-1)! \end{aligned} \tag{6}$$

Again, Equation 3 can be used to solve for the $k=2$ case in terms of the $k=1$ case. Here, the harmonic numbers start to be relevant, and as we will see, lead to an efficient representation of $\begin{bmatrix} n \\ k \end{bmatrix}$ in terms of the Harmonic Numbers:

$$\begin{aligned} \begin{bmatrix} n \\ 2 \end{bmatrix} &= (n-1)! \sum_{m=2}^{n} \frac{1}{(m-1)} \\ &= (n-1)! \sum_{m=1}^{n-1} \frac{1}{m} \\ &= (n-1)!\, H_{n-1} \end{aligned} \tag{7}$$

This sum identity can used with a proof by induction to attain higher k solutions. I have found up to $k=4$ in the literature. The solutions that I have found in the literature are:

$$\begin{aligned} \begin{bmatrix} n \\ 3 \end{bmatrix} &= \frac{(n-1)!}{2} \left(H_{n-1}^2 - H_{n-1}^{(2)} \right) \\ \begin{bmatrix} n \\ 4 \end{bmatrix} &= \frac{(n-1)!}{6} \left(H_{n-1}^3 - 3H_{n-1}H_{n-1}^{(2)} + 2H_{n-1}^{(3)} \right) \end{aligned} \tag{8}$$

The solutions I have worked out from induction are:

$$\begin{aligned} \begin{bmatrix} n \\ 5 \end{bmatrix} &= \frac{(n-1)!}{24} \left(H_{n-1}^4 + 8H_{n-1}H_{n-1}^{(3)} - 6H_{n-1}^2 H_{n-1}^{(2)} - 6H_{n-1}^{(4)} + 3\left(H_{n-1}^{(2)}\right)^2 \right) \\ \begin{bmatrix} n \\ 6 \end{bmatrix} &= \frac{(n-1)!}{120} \left(H_{n-1}^5 - 30H_{n-1}H_{n-1}^4 + 20H_{n-1}^2 H_{n-1}^{(3)} - 10H_{n-1}^{(2)} H_{n-1}^3 \right) \\ &\quad + \frac{(n-1)!}{120} \left(15\left(H_{n-1}^{(2)}\right)^2 H_{n-1} + 24H_{n-1}^{(5)} - 20H_{n-1}^{(2)} H_{n-1}^{(3)} \right) \\ \begin{bmatrix} n \\ 7 \end{bmatrix} &= \frac{(n-1)!}{720} \left(H_{n-1}^6 - 120H_{n-1}^{(6)} + 144H_{n-1}H_{n-1}^{(5)} - 90H_{n-1}^{(4)} H_{n-1}^2 \right) \\ &\quad + \frac{(n-1)!}{720} \left(90H_{n-1}^{(2)} H_{n-1}^{(4)} - 15H_{n-1}^{(2)} H_{n-1}^4 + 40\left(H_{n-1}^{(3)}\right)^2 + 40H_{n-1}^{(3)} H_{n-1}^3 \right) \\ &\quad + \frac{(n-1)!}{720} \left(45\left(H_{n-1}^{(2)}\right)^2 H_{n-1}^2 - 15\left(H_{n-1}^{(2)}\right)^3 - 120H_{n-1}^{(2)} H_{n-1}^{(3)} H_{n-1} \right) \end{aligned} \tag{9}$$

These expressions of the Stirling Numbers of the First Kind were found through assuming that the solution contained a series of all combinations of Harmonic Numbers that correspond to powers summing to $k-1$. The solution is scaled by a factor of $\frac{1}{(k-1)!}$. This scaling factor entails that the series of powers of the Harmonic numbers (and its higher order generalizations, namely $H_n^{(r)} =$

$\sum_{i=1}^{n} \frac{1}{k^r}$) must equal 1 when $n = k$, since the fraction of factorials cancels. This corresponds nicely to the behaviour that we expect to see from these numbers, since $\left[{n \atop n}\right] = 1$ for all positive integer n. Then, integer coefficients are applied to these series terms as a linear combination. The goal of the inductive proof is to solve for the coefficients that allow for the inductive step to be true. We will then show that the coefficients solved by the inductive step are also naturally the coefficients needed to verify the base case, that is that the series of powers of the Harmonic Numbers is 1 when $n = k$.

The relationships given in Equation 9 are remarkable in many respects. First off, they are relationships involving always non-integer Harmonic Numbers that are written as a linear combination with constant integer coefficients so that they always produce the integer result required by the Stirling Numbers of the First Kind. Secondly, there are patterns that arise that may allow for conjecture that may contribute to a complete set of expressions for any positive integer k (where clearly $n \geq k$). For example, the term H_{n-1}^{k-1} seems to always have coefficient equal to unity. The term $H_{n-1}^{(k-1)}$ seems to always have coefficient equal to $(-1)^{k-2}(k-2)!$. As well, the sum of the absolute value of the coefficients for a given solution seems to always equal $(k-1)!$, and the sum of the coefficients seems to always equal 0. There likely exists other such identities, but the problem with generalizing these ideas to the kth order Stirling Number is really that new combinations (and thus new coefficients) of Harmonic Numbers arise that have exponent powers that sum to $k-1$. It seems similar to a partitions problem in this respect, and as shown below, is strongly related to the binomial theorem during the inductive step.

Having made these notes, it is worth carrying out an example to illustrate the execution of the proof by induction for a relatively simple case. Take $\left[{n \atop 4}\right]$. The hypothesis step goes as follows:

$$\begin{aligned} \left[{n \atop 4}\right] &= (n-1)! \sum_{m=1}^{n-3} \frac{\left[{m+2 \atop 3}\right]}{(m+2)!} \\ &= \frac{(n-1)!}{2!} \sum_{m=1}^{n-3} \frac{H_{m+1}^2 - H_{m+1}^{(2)}}{(m+2)} \end{aligned} \tag{10}$$

The inductive step starts as follows:

$$\begin{aligned} \left[{n+1 \atop 4}\right] &= \frac{n!}{2!} \sum_{m=1}^{n-2} \frac{H_{m+1}^2 - H_{m+1}^{(2)}}{(m+2)} \\ &= \frac{n!}{2!} \left(\sum_{m=1}^{n-3} \frac{H_{m+1}^2 - H_{m+1}^{(2)}}{(m+2)} + \frac{H_{n-1}^2 - H_{n-1}^{(2)}}{n} \right) \\ &= n \left[{n \atop 4}\right] + \frac{(n-1)!}{2!} \left(H_{n-1}^2 - H_{n-1}^{(2)} \right) \\ &= n \left[{n \atop 4}\right] + \left[{n \atop 3}\right] \end{aligned} \tag{11}$$

which is a defining identity for the Stirling Numbers of the First Kind (illustrated in this case for $k = 4$). This identity is given on Wikipedia as a recursive relation for general k ($\left[{n+1 \atop k}\right] = n\left[{n \atop k}\right] + \left[{n \atop k-1}\right]$) Now we will show how to solve for the coefficients. We propose the following hypothesis:

$$\left[{n \atop 4}\right] = \frac{(n-1)!}{3!} \left(aH_{n-1}^3 + bH_{n-1}H_{n-1}^{(2)} + cH_{n-1}^{(3)} \right) \tag{12}$$

which is a linear combination of all possible combinations of Harmonic Numbers that sum to the power $k-1=3$. Now, we wish to show that:

$$\begin{aligned}
\begin{bmatrix} n+1 \\ 4 \end{bmatrix} &= n \begin{bmatrix} n \\ 4 \end{bmatrix} + \frac{n!}{2!} \frac{\left(H_{n-1}^2 - H_{n-1}^{(2)}\right)}{n} \\
&= \frac{n!}{3!}\left(aH_{n-1}^3 + bH_{n-1}H_{n-1}^{(2)} + cH_{n-1}^{(3)}\right) + \frac{n!}{2!}\frac{\left(H_{n-1}^2 - H_{n-1}^{(2)}\right)}{n} \\
&= \frac{n!}{3!}\left(aH_n^3 + bH_nH_n^{(2)} + cH_n^{(3)}\right)
\end{aligned} \tag{13}$$

where the second to third line is the critical step. Now we apply the following identities:

$$\begin{aligned}
H_{n-1} &= H_n - \frac{1}{n} \\
H_{n-1}^{(2)} &= H_n^{(2)} - \frac{1}{n^2} \\
H_{n-1}^{(3)} &= H_n^{(3)} - \frac{1}{n^3}
\end{aligned} \tag{14}$$

After substituting in the second line of Equation 13 and distributing the terms / making cancellations, we arrive at the result in the third line of Equation 13 *if* $a=1$, $b=-3$, and $c=2$.

This method can be applied for larger k by using linear algebra to write out a matrix for an overdetermined system of equations arising from the above method, where each "type" of term in the distributed calculation is represented by a row in the matrix. All of these rows must cancel with the proper solution to the integer coefficients as they aren't a part of the result in the third line of Equation 13. It is worthwhile to write out the entire overdetermined matrix to ensure that when the solution is found on inversion, all equations adhere and we have complete cancellation of each of the "extra terms". Furthermore, we write the third power terms from Equation 13 as the left side matrix A, where each column corresponds to a coefficient a, b, c... etc. in the solution expression. Then, the right side is comprised of a vector as the second power terms from Equation 13 - call it $\mathbf{w}$. Then, inverting A and applying it to the right side gives the solution vector $\mathbf{x}$:

$$\mathbf{x} = A^{-1}\mathbf{w} \tag{15}$$

All solutions found in Equation 9 were found through this linear algebra method. It seems nearly certain that higher solutions could be found through this method, though obviously keeping track of all terms arising from distributing all combinations of powers of Harmonic Numbers to the $k-1$ becomes a challenging but possible task.

The matrix A for the example above would be:

$$\begin{aligned}
A_4 &= \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \\
\mathbf{w}_4 &= \begin{bmatrix} -3 \\ 6 \\ -6 \\ 3 \end{bmatrix}
\end{aligned} \tag{16}$$

This concludes my first post on the blog forum.