Title: Finding Upper Bound Limits to Infinite Series
Author: Josh Myers

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At the end of the last post, I challenged the reader to show that an infinite series had an upper bound that was related to a variable in the equation. In this post, I will carry out the calculation and then discuss another example that I have found circulating social media recently. Furthermore, when the summation function does not vanish in a straightforward way at ∞, the value of the sum begins to depend on the limiting behaviour of the infinite upper bound. The value of the sum is typically infinity in this case - however, in the last post, a variable multiplying the sum was taken in the limit as it goes to 0, so the result had a finite value. This variable was used to express the limiting behaviour of the upper bound of the infinite sum.

From last time, we were calculating on the generating function of the Bernoulli Polynomials:

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n\left(x\right)\frac{t^n}{n!} \tag{1}$$

Both sides are 1 as $t\to 0$, so we work with this result. Expanding the denominator as a geometric series, we find:

$$\begin{aligned}1=\lim_{t\to 0}\frac{te^{t(x-1)}}{1-e^{-t}}&=\lim_{t\to 0}\left(t\sum_{n=0}^{\infty}e^{-nt}\right)\\&=\lim_{t\to 0}\frac{\sum_{n=0}^{\infty}e^{-nt}}{\frac{1}{t}}\\&=\lim_{t\to 0}\frac{\sum_{n=0}^{\infty}n^se^{-nt}}{s!\frac{1}{t^{s+1}}}\\&=\lim_{t\to 0}\lim_{N\to\infty}\frac{t}{N}\sum_{n=0}^{\infty}e^{-nt}\sum_{s=0}^{N}\frac{(nt)^s}{s!}\\&=\lim_{t\to 0}\lim_{N\to\infty}\frac{t}{N}\sum_{n=0}^{\infty}(1)\\&=\lim_{t\to 0}t^2\sum_{n=0}^{\infty}(1)\end{aligned} \tag{2}$$

Where these lines follow from L'Hopital's rule. The "∞" in the upper bound must now be specified. We know that the limiting result must be 1, so this means that the infinite sum must have about $\frac{1}{t^2}$ terms (as $t\to 0$). Thus the nature of the infinite upper bound is specified. However, it seems at least possible that the geometric series always went to infinity as $\frac{1}{t^2}$ as $t\to 0$, as the fact that the limiting result must be equal to 1 seems somewhat unrelated to the nature of the upper bound of the series expansion. This statement is not to imply that all geometric series have this upper bound limit as this geometric series is special in that the exponent has a term who's limit is going to 0. Indeed, it is quite likely that for most instances of the geometric series (where the terms vanish as $n \to \infty$), the nature of the upper bound limit is irrelevant. But this example is one important instance where the limiting nature of the upper bound matters.

This result motivates insightful philosophical discussion, including that if the sum upper bound always had limiting behaviour on the order of $\frac{1}{t^2}$ as $t \to 0$, then wouldn't taking the derivative with respect to t of the sum for L'Hopital's rule be leaving out the t-dependence of the upper bound? The idea of taking the derivative with respect to an upper bound of a sum is indeed troubling, but perhaps we can reconcile these problems by claiming that only when the $t \to 0$ is the upper bound going like $\frac{1}{t^2}$ and it only does so from an integer perspective, rather than in a continuous way. This then brings up the important idea that perhaps the geometric sum only has the upper bound t-dependence when $t \to 0$ - it certainly seems to be the only case where the limiting behaviour of the upper bound of the sum seems to matter, as the terms do not necessarily vanish as $n \to \infty$.

I will demonstrate another example where the limiting behaviour of an upper bound arises in an important way. Recently Fermat's Library posted on LinkedIn the following statement:

$$\left(\frac{1}{10^5}\sum_{n=-\infty}^{\infty} e^{-\frac{n^2}{10^{10}}}\right)^2 \approx \pi \tag{3}$$

Then Fermat's Library remarks that the sum given calculates π to the first 42 billion digits. While this seems impressive, the result is still computationally an infinite sum, and is only a special case of the exact relation given below:

$$\left(\lim_{\beta\to 0^+}\sqrt{\beta}\sum_{n=-\infty}^{\infty} e^{-\beta n^2}\right)^2 = \left(\sqrt{\beta}\int_{-\infty}^{\infty} \mathrm{d}x e^{-\beta x^2}\right)^2 = \left(\sqrt{\beta}\sqrt{\frac{\pi}{\beta}}\right)^2 = \pi \tag{4}$$

where the first equality is a result of the Euler-Maclaurin formula. Let's zoom in on the relation on the very left hand side of Equation 4, and use L'Hopital's rule again:

$$\begin{aligned}
\sqrt{\pi} &= \lim_{\beta\to 0}\lim_{N\to\infty}\frac{\sqrt{\beta}}{N}\sum_{n=-\infty}^{\infty} e^{-\beta n^2}\sum_{s=0}^{N}\frac{\left(2\beta n^2\right)^s}{(2s-1)!!} \\
&= \lim_{\beta\to 0}\left(\sqrt{\beta}\right)^3\sum_{n=-\infty}^{\infty} e^{-\beta n^2} F_{11}\left(1;\frac{1}{2};\beta n^2\right) \\
&= \sqrt{\pi}\lim_{\beta\to 0}\beta^2\sum_{n=-\infty}^{\infty} n\,\mathrm{erf}\left(\sqrt{\beta}n\right) \\
&= \sqrt{\pi}\lim_{\beta\to 0}2\beta^2\sum_{n=0}^{\infty} n\,\mathrm{erf}\left(\sqrt{\beta}n\right)
\end{aligned} \tag{5}$$

where $(2s-1)!! = \frac{(2s)!}{2^s s!}$ is the double factorial, erf is the error function, and F_{11} is the confluent hypergeometric function. We assume that the doubler infinite sum goes to $\pm\infty$ in the same way for both the top and the bottom of the sum. It turns out that this sum has upper bound limiting behaviour that goes like $\frac{1}{\beta}$ as $\beta \to 0$ - I have confirmed this for myself numerically. I will strive to prove it analytically in a future post. This concludes this blog post.