Title: Ideas Related to Fourier Series PDE
Author: Josh Myers

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Consider the complex Fourier Coefficient:

$$\begin{aligned} c_n &= \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}x'\, g\left(x'\right) e^{-2\pi i n \frac{x'}{L}} \\ \implies g\left(x\right) &= \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n \frac{x}{L}} = c_0 + \sum_{n\neq 0} c_n e^{2\pi i n \frac{x}{L}} \end{aligned} \tag{1}$$

Previously on this site, I have discussed about the following PDE:

$$c_n = \left(-1\right)^n \frac{g\left(\frac{L}{2}\right) + g\left(-\frac{L}{2}\right)}{2} - L\frac{\partial c_n}{\partial L} - n\frac{\partial c_n}{\partial n} \tag{2}$$

Now consider the case where $g\left(x\right) = g\left(-x\right)$ is even, and $\frac{L}{2} = x$ is a dummy variable. In this case, we find that c_n is also even:

$$\begin{aligned} c_n &= \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}x'\, g\left(-x'\right) e^{-2\pi i n \frac{x'}{L}} \\ &= \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}x'\, g\left(x'\right) e^{2\pi i n \frac{x'}{L}} \\ &= c_{-n} \end{aligned} \tag{3}$$

Now consider the case where $g\left(x\right) = -g\left(-x\right)$ is odd. In this case, we find that c_n is odd:

$$\begin{aligned} c_n &= -\frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}x'\, g\left(-x'\right) e^{-2\pi i n \frac{x'}{L}} \\ &= -\frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}x'\, g\left(x'\right) e^{2\pi i n \frac{x'}{L}} \\ &= -c_{-n} \end{aligned} \tag{4}$$

Now consider $\frac{L}{2} = x$, such that $g\left(x\right) = 0$ for odd functions. Thus, without loss of generality, we will consider the even case only. Inserting Equation 2 into Equation 1, we get:

$$\begin{aligned} g\left(x\right) &= c_0\left(x\right) + 2\sum_{n=1}^{\infty} c_n\left(x\right)\left(-1\right)^n \\ &= c_0\left(x\right) + 2\sum_{n=1}^{\infty}\left(g\left(x\right) - \left(\left(-1\right)^n\left(n\frac{\partial c_n}{\partial n}\left(x\right) + x\frac{\partial c_n}{\partial x}\left(x\right)\right)\right)\right) \end{aligned} \tag{5}$$

where:

$$c_n(x) = \frac{1}{2x}\int_{-x}^{x} \mathrm{d}x'\, g(x')\, e^{-\pi i n \frac{x'}{x}} \tag{6}$$

Re-arranging Equation 5, we get:

$$\begin{aligned} g(x) &= \lim_{N\to\infty} \frac{2\sum_{n=1}^{N}\left((-1)^n\left(n\frac{\partial c_n}{\partial n}(x) + x\frac{\partial c_n}{\partial x}(x)\right)\right) - c_0(x)}{2N-1} \\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left((-1)^n\left(n\frac{\partial c_n}{\partial n}(x) + x\frac{\partial c_n}{\partial x}(x)\right)\right) \end{aligned} \tag{7}$$

Analytically computing the two derivatives in Equation 7, we get:

$$\begin{aligned} x\frac{\partial c_n}{\partial x} &= x\partial_x \frac{1}{2}\int_{-1}^{1}\mathrm{d}x'\, g(xx')\, e^{-\pi i n x'} \\ &= \frac{1}{2}\int_{-x}^{x}\mathrm{d}x'\, x' g(x')\, e^{-\pi i n \frac{x'}{x}} \\ &= \frac{1}{2}\int_{-x}^{x}\mathrm{d}x'\, \tilde{g}(x')\, e^{-\pi i n \frac{x'}{x}} \\ &= x\,\tilde{c}_n(x) \\ n\frac{\partial c_n}{\partial n} &= n\,\partial_n \frac{1}{2x}\int_{-x}^{x}\mathrm{d}x'\, g(x')\, e^{-\pi i n \frac{x'}{x}} \\ &= \frac{n}{2x}\int_{-x}^{x}\mathrm{d}x'\, g(x')\left(\frac{-i\pi x'}{x}\right) e^{-\pi i n \frac{x'}{x}} \\ &= \frac{1}{2x^2}\int_{-x}^{x}\mathrm{d}x'\, g(x')\,(-i\pi n x')\, e^{-\pi i n \frac{x'}{x}} \\ &= \frac{-i\pi n}{x}\tilde{c}_n(x) \end{aligned} \tag{8}$$

It is easy to see that $x\tilde{c}_n(x)$ should sum in n to a finite series result in terms of finite x, as it is simply a factor of x different from the positive index Fourier Series of another (related) function. This implies that dividing this finite result by the limit variable N should lead to a 0 result. However, $\frac{-i\pi n}{x}\tilde{c}_n(x)$ is not as simple to compute, as an additional factor of n outside of $\tilde{c}_n(x)$ exists to change the series outcome. Thus, given that we've computed correctly, we attain the following satisfying result:

$$g(x) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}(-1)^n\, n\frac{\partial c_n}{\partial n}(x) \tag{9}$$

which holds for an even $g(x)$ that can be written as a Fourier Series. I have confirmed this result numerically. An interesting side note is that if we take the bottom half of Equation 8 and sum it in n, we get:

$$\begin{aligned} g(x) &= \frac{i\pi}{2x^2}\int_{-x}^{x}\mathrm{d}x'\, g(x')\, x' \lim_{N\to\infty}\sum_{n=1}^{N}(-1)^{n-1}\, n e^{-\pi i n \frac{x'}{x}} \\ \implies x\delta\left(\frac{x'}{x}\right) &= \frac{i\pi}{2}\frac{x'}{x}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}(-1)^{n-1}\, n e^{-\pi i n \frac{x'}{x}} \end{aligned} \tag{10}$$

where we have found a Dirac-Delta *like* (but not the same numerically) function. This concludes this blog post.