Title: Evaluating a Two State Degenerate System
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In the previous post, it was found that for a two-state, time dependent, degenerate system, the probability amplitudes abide by the following ordinary differential equations:

$$\begin{aligned}
&\frac{\partial c_1(t)}{\partial t} + c_1(t)\langle\psi_1(\mathbf{r},t)|\frac{\partial}{\partial t}|\psi_1(\mathbf{r},t)\rangle + c_2(t)\langle\psi_1(\mathbf{r},t)|\frac{\partial}{\partial t}|\psi_2(\mathbf{r},t)\rangle = 0 \\
&\frac{\partial c_2(t)}{\partial t} + c_1(t)\langle\psi_2(\mathbf{r},t)|\frac{\partial}{\partial t}|\psi_1(\mathbf{r},t)\rangle + c_2(t)\langle\psi_2(\mathbf{r},t)|\frac{\partial}{\partial t}|\psi_2(\mathbf{r},t)\rangle = 0
\end{aligned} \tag{1}$$

where it was found in the last post that $\langle\psi_j(\mathbf{r},t)|\frac{\partial}{\partial t}|\psi_j(\mathbf{r},t)\rangle$ is purely imaginary for integer j. Now, AI was able to find a two-state system with the time-dependent, degenerate property - it specified the following system:

$$\begin{aligned}
\hat{H}(t) &= E(t)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
|\psi_1(t)\rangle &= \begin{bmatrix} \cos(\omega t) \\ \sin(\omega t) \end{bmatrix},\ |\psi_2(t)\rangle = \begin{bmatrix} -\sin(\omega t) \\ \cos(\omega t) \end{bmatrix}
\end{aligned} \tag{2}$$

where $\hat{H}(t)$ is written in the energy eigenbasis. To see that these definitions are consistent, we will first note:

$$|\Psi(t)\rangle = c_1(t)|\psi_1(t)\rangle e^{-\frac{i\int_{t_0}^{t}\mathrm{d}\tilde{t}\,E(\tilde{t})}{\hbar}} + c_2(t)|\psi_2(t)\rangle e^{-\frac{i\int_{t_0}^{t}\mathrm{d}\tilde{t}\,E(\tilde{t})}{\hbar}}$$

and this satisfies the time-dependent Schrodinger Equation:

$$\begin{aligned}
\hat{H}(t)|\Psi(t)\rangle &= i\hbar\frac{\partial|\Psi(t)\rangle}{\partial t} \\
&= i\hbar\left(\frac{\partial c_1(t)}{\partial t}|\psi_1(t)\rangle e^{-\frac{i\int_{t_0}^{t}\mathrm{d}\tilde{t}\,E(\tilde{t})}{\hbar}} + \omega c_1(t)|\psi_2(t)\rangle e^{-\frac{i\int_{t_0}^{t}\mathrm{d}\tilde{t}\,E(\tilde{t})}{\hbar}}\right) \\
&\quad + i\hbar\left(\frac{\partial c_2(t)}{\partial t}|\psi_2(t)\rangle e^{-\frac{i\int_{t_0}^{t}\mathrm{d}\tilde{t}\,E(\tilde{t})}{\hbar}} - \omega c_2(t)|\psi_1(t)\rangle e^{-\frac{i\int_{t_0}^{t}\mathrm{d}\tilde{t}\,E(\tilde{t})}{\hbar}}\right) \\
&\quad + E(t)\left(c_1(t)|\psi_1(t)\rangle e^{-\frac{i\int_{t_0}^{t}\mathrm{d}\tilde{t}\,E(\tilde{t})}{\hbar}} + c_2(t)|\psi_2(t)\rangle e^{-\frac{i\int_{t_0}^{t}\mathrm{d}\tilde{t}\,E(\tilde{t})}{\hbar}}\right)
\end{aligned} \tag{3}$$

This time-dependent equation is solved by the following observation, independent of the time-dependent form of the $E(t)$:

$$\frac{\partial c_1(t)}{\partial t} = \omega c_2(t),\ \frac{\partial c_2(t)}{\partial t} = -\omega c_1(t)$$

Thus, implementing these observations for Equation 1:

$$\begin{aligned}\omega c_2(t) + \omega c_1(t)\langle\psi_2(t)|\psi_1(t)\rangle - \omega c_2(t)\langle\psi_1(t)|\psi_1(t)\rangle = \omega c_2(t) - \omega c_2(t) = 0\\ -\omega c_1(t) + \omega c_1(t)\langle\psi_2(t)|\psi_2(t)\rangle - \omega c_2(t)\langle\psi_2(t)|\psi_1(t)\rangle = -\omega c_1(t) + \omega c_1(t) = 0\end{aligned} \tag{4}$$

And thus this system is consistent with the form of Equation 1. We note that this is a special case where the degenerate states are naturally orthogonal, and also that $\langle\psi_j(t)|\frac{\partial}{\partial t}|\psi_j(t)\rangle$ is still strictly speaking, imaginary, but in this case is 0 for every integer j. This serves as a comforting confirmation that the resultant calculations from the last post may be airing on the side of correctness. Now, before finishing this short post, we will show the nature of calculations for correcting for a two state degenerate subspace where the two states are not naturally orthogonal. This procedure is called the Gram-Schmidt process, and it maintains degeneracy in the transformed states that was present in the original states. Given two degenerate, normalized states $|\phi_1\rangle, |\phi_2\rangle$, we can construct two states that span the same basis (denoted as $|\psi_1\rangle, |\psi_2\rangle$ by:

$$\begin{aligned}&|\psi_1\rangle = |\phi_1\rangle\\ &|\psi_2\rangle = |\phi_2\rangle - \langle\psi_1|\phi_2\rangle|\psi_1\rangle\\ \implies &\langle\psi_1|\psi_2\rangle = 0,\ \langle\psi_1|\psi_1\rangle = 1,\ \langle\psi_2|\psi_2\rangle = \langle\psi_2|\phi_2\rangle = 1 - |\langle\psi_1|\phi_2\rangle|^2\\ \implies &\hat{H}|\psi_1\rangle = E|\psi_1\rangle,\ \hat{H}|\psi_2\rangle = E|\psi_2\rangle\end{aligned} \tag{5}$$

and thus the states are now orthogonal, while still remaining degenerate with the same energy. All that is left to do is normalize $|\psi_2\rangle$ by manually dividing the state by $1 - |\langle\psi_1|\phi_2\rangle|^2$ and then inserting the two normalized, orthogonal, and degenerate subspace basis vectors back into the quantum superposition sum and carry out calculations as typical. Thus, we can always manage to work with orthonormal eigenstates if we choose. The Gram-Schmidt process is developed in sequence as more degenerate basis vectors arise - for N degenerate, non-orthogonal eigenstates:

$$\begin{aligned}|\psi_j\rangle = |\phi_j\rangle - \sum_{i=1}^{j-1}\langle\psi_{i-1}|\phi_j\rangle|\psi_{i-1}\rangle\\ \implies \langle\psi_j|\psi_j\rangle = 1 - \sum_{i=1}^{j-1}|\langle\psi_{i-1}|\phi_j\rangle|^2\end{aligned} \tag{6}$$

I will conclude this post by showing that for two normalized states $|\phi_i\rangle, |\phi_j\rangle$, the inner product of those states has an absolute value that is at most 1:

$$\langle\phi_i|\phi_j\rangle = \sqrt{\langle\phi_i|\phi_i\rangle}\sqrt{\langle\phi_j|\phi_j\rangle}\cos(\theta_{ij}) = \cos(\theta_{ij}) \tag{7}$$

where θ_{ij} is the angle between $\langle\phi_i|$ and $|\phi_j\rangle$ in the Hilbert Space. In my next post, I will examine the angular momentum and spin quantum calculations in a magnetic field. This concludes this blog post.