Title: Solution to Power Sum by Euler-Maclaurin Formula
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This post will derive the solution to the famous power sum, first developed by Jacob Bernoulli. In my previous post, I outlined the Euler-Maclaurin formula, developed after Bernoulli's discovery. However, the Euler-Maclaurin formula provides a highly intuitive route to Bernoulli's result, and thus will be covered here. Bernoulli discovered:

$$\sum_{k=1}^{n} k^p = n^p + \sum_{k=0}^{p} \frac{B_k\, p!}{k!\,(p-k+1)!} n^{p-k+1} \tag{1}$$

Recall the Euler-Maclaurin formula:

$$\begin{gathered}\sum_{i=j}^{q} f(i) = \int_j^q \mathrm{d}x\, f(x) + \sum_{r=1}^{\lfloor \frac{s}{2} \rfloor} \frac{B_{2r}}{(2r)!} \left(f^{(2r-1)}(q) - f^{(2r-1)}(j) \right) \\ \frac{f(q)+f(j)}{2} + (-1)^p \int_j^q \mathrm{d}x\, \frac{B_s(x)}{s!} f^{(s)}(x)\end{gathered} \tag{2}$$

In this case, $j = 1$, $q = n$, $f(x) = x^p$, and $f^{(s)}(x) = \frac{p!}{(p-s)!} x^{p-s}$. Because of the nature of $f(x)$, $f^{(s)}(x) = 0$ for $s > p$ - thus, because s is a free parameter in the formula, we can choose that $s = p+1$. Also, since $B_{2r-1} = 0$ for $r > 1$, we can generalize the sum in r to both even and odd positive integers. From this, we attain:

$$\sum_{k=1}^{n} k^p = \int_1^n \mathrm{d}x\, x^p + \frac{n^p+1}{2} + \sum_{k=2}^{p} \frac{B_k}{k!} \frac{p!}{(p-k+1)!} \left(n^{p-k+1} - 1 \right) \tag{3}$$

Manipulating:

$$\begin{aligned}\sum_{k=1}^{n} k^p &= \frac{1}{p+1} \left(n^{p+1} - 1 \right) + \frac{n^p+1}{2} + \sum_{k=2}^{p} \frac{B_k}{k!} \frac{p!}{(p-k+1)!} \left(n^{p-k+1} - 1 \right) \\ &= n^p + \sum_{k=0}^{p} \frac{B_k}{k!} \frac{p!}{(p-k+1)!} \left(n^{p-k+1} - 1 \right) \\ &= n^p + \sum_{k=0}^{p} \frac{B_k}{k!} \frac{p!}{(p-k+1)!} n^{p-k+1} - \sum_{k=0}^{p} \frac{B_k}{k!} \frac{p!}{(p-k+1)!}\end{aligned} \tag{4}$$

The last term in this Equation 4 must vanish to attain the result in Equation 1. It turns out that this equation is a defining result of the Bernoulli number, as can be found on the Wikipedia page on the topic. As given on Wikipedia:

$$\begin{aligned} \sum_{k=0}^{m} \binom{m+1}{k} B_k &= \delta_{m0} \\ (m+1)\sum_{k=0}^{m} \binom{m}{k} \frac{B_k}{m-k+1} &= \delta_{m0} \\ (m+1)\sum_{k=0}^{m} \frac{B_k}{k!} \frac{m!}{(m-k+1)!} &= \delta_{m0} \end{aligned} \tag{5}$$

The last term in Equation 4 has the same form as the last line of Equation 5, and thus it vanishes in every case except when $p = 0$ (in which case it equals 1), as expected. Then, we have the result touted in Equation 1. This concludes this (short) blog post. This post leads naturally into an analysis of the generating function of the Bernoulli Polynomials, but is too long to start and complete here.