Title: Re-examination of Stieltjes Constants
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The Stieltjes Constants are typically defined as:

$$\gamma_n = \lim_{m\to\infty} \sum_{k=1}^{m} \frac{\ln^n(k)}{k} - \frac{\ln^{n+1}(m)}{n+1} \tag{1}$$

In the last Stieltjes post, I found the following relation:

$$\gamma_n = \lim_{m\to\infty} \sum_{q=0}^{n} \frac{\binom{n}{q}}{n!} (-1)^{n-q} q^n \int_0^1 \mathrm{d}t \, \frac{1-t^m}{1-t} \lim_{z\to 0} \partial_z^n \left(\frac{(-\ln(t))^{-z}}{\Gamma(1-z)} \right) - \frac{\ln^{n+1}(m)}{n+1} \tag{2}$$

Evaluating the sum in q:

$$\sum_{q=0}^{n} \frac{\binom{n}{q}}{n!} (-1)^{n-q} q^n = 1$$

Making the second term an integral representation:

$$\frac{\ln^{n+1}(m)}{n+1} = (-1)^{n+1} \frac{\ln^{n+1}\left(\frac{1}{m}\right)}{n+1} = (-1)^n \int_{\frac{1}{m}}^{1} \frac{\ln^n(t)}{t}$$

To calculate the nth derivative in z, we calculate:

$$\begin{aligned} \left(\partial_z^p \left((-\ln(t))^{-z}\right)\right)_{z\to 0} &= (-\ln(-\ln(t)))^p \\ \left(\partial_z^r \left(\frac{1}{\Gamma(1-z)}\right)\right)_{z\to 0} &= B_r\left(\psi_0(1), -\psi_1(1), ..., (-1)^{n-1}\psi_{n-1}(1)\right) \end{aligned} \tag{3}$$

where $B_r(x_0, x_1, ..., x_{n-1})$ is the complete exponential Bell polynomials:

$$B_r\left(\psi_0(1), -\psi_1(1), ..., (-1)^{n-1}\psi_{n-1}(1)\right) = \sum_{\nu=1}^{r} B_{r,\nu}\left(\psi_0(1), -\psi_1(1), ..., (-1)^{n-1}\psi_{n-1}(1)\right)$$

where $B_{n,k}(x_0, x_1, ..., x_{n-1})$ are the incomplete Bell polynomials. The complete exponential Bell polynomials have the following generating function:

$$\exp\left(\sum_{a=1}^{\infty} x_a \frac{t^a}{a!}\right) = 1 + \sum_{b=1}^{\infty} B_b(x_1, x_2, ..., x_b) \frac{t^b}{b!}$$

So, the nth derivative in Equation 2 is:

$$\begin{aligned}\lim_{z\to0}\partial_z^n\left(\frac{(-\ln(t))^{-z}}{\Gamma(1-z)}\right)&=\lim_{z\to0}\sum_{\omega=0}^{n}\binom{n}{\omega}\partial_z^{n-\omega}\left((-\ln(t))^{-z}\right)\partial_z^{\omega}\left(\frac{1}{\Gamma(1-z)}\right)\\&=\sum_{\omega=0}^{n}\binom{n}{\omega}(-\ln(-\ln(t)))^{n-\omega}B_\omega\left((-1)^0\psi_0(1),...,(-1)^{n-1}\psi_{n-1}(1)\right)\end{aligned}\tag{4}$$

All of this work leads to the following restatement of Equation 2:

$$\begin{aligned}\gamma_n=\lim_{m\to\infty}-\int_{\frac{1}{m}}^{1}\mathrm{d}t\,\frac{(-\ln(t))^n}{t}+\int_0^1\mathrm{d}t\\\sum_{\omega=0}^{n}\left(\binom{n}{\omega}\frac{1-t^m}{1-t}(-\ln(-\ln(t)))^{n-\omega}B_\omega\left((-1)^0\psi_0(1),...,(-1)^{n-1}\psi_{n-1}(1)\right)\right)\end{aligned}\tag{5}$$

Now, because we have two terms going to ∞ that we subtract to give a finite answer, we cannot simply use L'Hopital's Rule with a ratio of those terms because the finite answer part will be lost during differentiation. The most advisable path is to series expand the two terms in m, cancel the divergent parts, and see what remains. Taking a different approach to my previous post, we find that:

$$\begin{aligned}\frac{\ln^{n+1}(m)}{n+1}&=\int_{\frac{1}{m}}^{1}\mathrm{d}t\,\frac{(-\ln(t))^n}{t}\\&=(-1)^{n-1}n\int_{\frac{1}{m}}^{1}\mathrm{d}t\int_t^1\mathrm{d}t'\,\frac{(\ln(t'))^{n-1}}{t't}\\&=n!\int_{\frac{1}{m}}^{1}\frac{\mathrm{d}t_0}{t_0}\prod_{\mu=1}^{n-1}\left(\int_{t_{\mu-1}}^{1}\mathrm{d}t_\mu\frac{1}{t_\mu}\right)(-\ln(t_{n-1}))\\&=n!\int_{\frac{1}{m}}^{1}\frac{\mathrm{d}t_0}{t_0}\prod_{\mu=1}^{n-1}\left(\int_{t_{\mu-1}}^{1}\mathrm{d}t_\mu\frac{1}{t_\mu}\right)\sum_{s=1}^{\infty}\frac{(1-t_{n-1})^s}{s}\\&=n!\sum_{s=1}^{\infty}\frac{1}{s}\int_{\frac{1}{m}}^{1}\frac{\mathrm{d}t_0}{t_0}\prod_{\mu=1}^{n-2}\left(\int_{t_{\mu-1}}^{1}\mathrm{d}t_\mu\frac{1}{t_\mu}\right)\int_0^{1-t_{n-2}}\mathrm{d}t_{n-1}\frac{t_{n-1}^s}{1-t_{n-1}}\\&=n!\sum_{s=1}^{\infty}\frac{1}{s}\int_{\frac{1}{m}}^{1}\frac{\mathrm{d}t_0}{t_0}\prod_{\mu=1}^{n-2}\left(\int_{t_{\mu-1}}^{1}\mathrm{d}t_\mu\frac{1}{t_\mu}\right)B_{1-t_{n-2}}(s+1,0)\\&=n!\sum_{s=1}^{\infty}\frac{1}{s}\int_{\frac{1}{m}}^{1}\frac{\mathrm{d}t_0}{t_0}\prod_{\mu=1}^{n-3}\left(\int_{t_{\mu-1}}^{1}\mathrm{d}t_\mu\frac{1}{t_\mu}\right)\sum_{w_1=0}^{\infty}\frac{1}{w_1+s+1}\int_0^{1-t_{n-3}}\mathrm{d}t_{n-2}\frac{t_{n-2}^{w_1+s+1}}{1-t_{n-2}}\\&=n!\sum_{s=1}^{\infty}\frac{1}{s}\int_{\frac{1}{m}}^{1}\frac{\mathrm{d}t_0}{t_0}\prod_{\mu=1}^{n-3}\left(\int_{t_{\mu-1}}^{1}\mathrm{d}t_\mu\frac{1}{t_\mu}\right)\sum_{w_1=0}^{\infty}\frac{1}{w_1+s+1}\sum_{w_2=0}^{\infty}\frac{(1-t_{n-3})^{w_2+w_1+s+2}}{w_2+w_1+s+2}\\&=n!\sum_{s=1}^{\infty}\frac{1}{s}\int_{\frac{1}{m}}^{1}\frac{\mathrm{d}t_0}{t_0}\prod_{\mu=1}^{n-1}\sum_{w_\mu=0}^{\infty}\frac{(1-t_0)^{s+n-1+\sum_{d=1}^{n-1}w_d}}{s+\mu+\sum_{d=1}^{\mu}w_d}\\&=n!\prod_{\mu=0}^{n}\sum_{w_\mu=\delta_{\mu0}}^{\infty}\frac{\left(1-\frac{1}{m}\right)^{n+\sum_{d=0}^{n}w_d}}{\mu+\sum_{d=0}^{\mu}w_d}\end{aligned}\tag{6}$$

where $\delta_{\mu 0}$ is the Kronecker Delta function. This is "simplified" by considering the Sterling Numbers of the First Kind explicitly - however, I will stick with this definition for now. Alternatively, to series expand the other term in Equation 5 is far simpler:

$$t^m = \sum_{\ell=0}^{\infty} \frac{m^\ell \ln^\ell (t)}{\ell!} \tag{7}$$

Now, revisiting Equation 5, we have:

$$\begin{aligned}\gamma_n = \lim_{m\to\infty} -n! \prod_{\mu=0}^{n} \sum_{w_\mu=\delta_{\mu 0}}^{\infty} \frac{\left(1-\frac{1}{m}\right)^{n+\sum_{d=0}^{n} w_d}}{\mu + \sum_{d=0}^{\mu} w_d} - \int_0^1 \mathrm{d}t \\ \sum_{\omega=0}^{n} \left(\binom{n}{\omega} \sum_{\ell=1}^{\infty} \frac{m^\ell \ln^\ell (t)}{(1-t)\,\ell!} \left(-\ln\left(-\ln(t)\right)\right)^{n-\omega} B_\omega \left((-1)^0 \psi_0(1), ..., (-1)^{n-1} \psi_{n-1}(1) \right) \right)\end{aligned} \tag{8}$$

Now attempting the integral, let $u = -\ln(t)$, thus $\mathrm{d}t = e^{-u}\mathrm{d}u$. Thus:

$$\begin{aligned}\int_0^1 \mathrm{d}t \, \frac{\ln^\ell (t) \ln^{n-\omega} \left(-\ln(t)\right)}{1-t} &= (-1)^\ell \int_0^\infty \mathrm{d}u \, \frac{e^{-u} u^\ell \ln^{n-\omega}(u)}{1-e^{-u}} \\ &= (-1)^\ell \sum_{\sigma=0}^{\infty} \int_0^\infty \mathrm{d}u \, e^{-(\sigma+1)u} u^\ell \ln^{n-\omega}(u) \\ &= (-1)^\ell \sum_{\sigma=0}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}\chi \, e^{-(\sigma+1)e^\chi} e^{(\ell+1)\chi} \chi^{n-\omega}\end{aligned} \tag{9}$$

The following remain open questions:

- how to evaluate the integral in Equation 9?
- how to evaluate the sums in ω of the complete exponential Bell Polynomials?
- how to cancel the divergent parts of the two terms of Equation 8 in terms of powers of m?

Indeed, understanding the first two question will greatly help with understanding the third question. In particular, the integral in terms of t will be the topic of immediate future investigation. Another potential point of improvement is to evaluate if $\ln^{n+1}(m)$ can be written as a first order simplification in terms of $\frac{1}{m}$. All to come in the future. Happy Canada Day 2026! This concludes this blog post.