Title: Lobachevsky's Formula and Fourier Transform
Author: Josh Myers

June 30, 2025

The first part of this post includes a proof and associated identities for Lobachevsky's Integral Formula. Then the second part highlights an application of Lobachevsky's formula to computing particular values of a Fourier Transform. First, consider the integral:

$$\int_0^{\infty} \mathrm{d}x \frac{\sin(x)}{x} f(x) \tag{1}$$

We can rewrite this integral formula as:

$$\int_0^{\infty} \mathrm{d}x \frac{\sin(x)}{x} f(x) = \sum_{k=1}^{\infty} \left(\int_{(2k)\frac{\pi}{2}}^{(2k+1)\frac{\pi}{2}} \mathrm{d}x + \int_{(2k-1)\frac{\pi}{2}}^{(2k)\frac{\pi}{2}} \mathrm{d}x + \int_0^{\frac{\pi}{2}} \mathrm{d}x \right) \left(\frac{\sin(x)}{x} f(x) \right) \tag{2}$$

Now we make a substitution of $x = t + k\pi$ for the first integral and $x = k\pi - t$ for the second integral. Then we stipulate that $f(x)$ must be periodic such that $f(x) = f(\pi + x)$ and $f(x) = f(\pi - x)$. Such a $f(x)$ is $f(x) = \cos(2x)$, $f(x) = \sin^2(x)$ or $f(x) = 1$. This gives:

$$\begin{aligned}
\int_0^{\infty} \mathrm{d}x \frac{\sin(x)}{x} f(x) &= \sum_{k=0}^{\infty} \mathrm{d}t \frac{\sin(t + k\pi)}{t + k\pi} f(t + k\pi) + \sum_{k=1}^{\infty} \int_0^{\frac{\pi}{2}} \mathrm{d}t \frac{\sin(k\pi - t)}{k\pi - t} f(k\pi - t) \\
&= \int_0^{\frac{\pi}{2}} \mathrm{d}t \left(\frac{\sin(t)}{t} f(t) + \sum_{k=1}^{\infty} (-1)^k \left(\frac{\sin(t)}{t + k\pi} f(t) + \frac{\sin(t)}{t - k\pi} f(t) \right) \right) \\
&= \int_0^{\frac{\pi}{2}} \mathrm{d}t \sin(t) f(t) \left(\frac{1}{t} + \sum_{k=1}^{\infty} (-1)^k \left(\frac{1}{t + k\pi} + \frac{1}{t - k\pi} \right) \right)
\end{aligned} \tag{3}$$

Now the magic happens. It is known:

$$\begin{aligned}
\frac{1}{\sin(t)} &= \frac{1}{t} + \sum_{k=1}^{\infty} (-1)^k \left(\frac{1}{t + k\pi} + \frac{1}{t - k\pi} \right) \\
\frac{1}{\sin^2(t)} &= \frac{1}{t^2} + \sum_{k=1}^{\infty} \left(\frac{1}{(t + k\pi)^2} + \frac{1}{(t - k\pi)^2} \right)
\end{aligned} \tag{4}$$

Proof of these formulas is given in Jolany 2018 (see https://ems.press/content/serial-article-files/45613). In any case, we see that the factor in the third line of Equation 3 is the same as the first line of Equation 4, so we get:

$$\int_0^{\infty} \mathrm{d}x \frac{\sin(x)}{x} f(x) = \int_0^{\frac{\pi}{2}} \mathrm{d}t f(t) \tag{5}$$

Jolany 2018 does more analysis, attaining the following formulas:

$$\begin{aligned}
\int_0^\infty \mathrm{d}x \frac{\sin^2(x)}{x^2} f(x) &= \int_0^{\frac{\pi}{2}} \mathrm{d}t f(t) \\
\int_0^\infty \mathrm{d}x \frac{\sin^4(x)}{x^4} f(x) &= \int_0^{\frac{\pi}{2}} \mathrm{d}t f(t) - \frac{2}{3}\int_0^{\frac{\pi}{2}} \mathrm{d}t \sin^2(t) f(t) \\
\int_0^\infty \mathrm{d}x \frac{\sin^6(x)}{x^6} f(x) &= \int_0^{\frac{\pi}{2}} \mathrm{d}t f(t) - \int_0^{\frac{\pi}{2}} \mathrm{d}t \sin^2(t) f(t) + \frac{2}{15}\int_0^{\frac{\pi}{2}} \mathrm{d}t \sin^4(t) f(t)
\end{aligned} \tag{6}$$

Setting $f(x) = 1$, we get:

$$\begin{aligned}
\int_0^\infty \mathrm{d}x \frac{\sin^2(x)}{x^2} f(x) &= \int_0^{\frac{\pi}{2}} \mathrm{d}t = \frac{\pi}{2} \\
\int_0^\infty \mathrm{d}x \frac{\sin^4(x)}{x^4} f(x) &= \int_0^{\frac{\pi}{2}} \mathrm{d}t - \frac{2}{3}\int_0^{\frac{\pi}{2}} \mathrm{d}t \sin^2(t) = \frac{\pi}{3} \\
\int_0^\infty \mathrm{d}x \frac{\sin^6(x)}{x^6} f(x) &= \int_0^{\frac{\pi}{2}} \mathrm{d}t - \int_0^{\frac{\pi}{2}} \mathrm{d}t \sin^2(t) + \frac{2}{15}\int_0^{\frac{\pi}{2}} \mathrm{d}t \sin^4(t) = \frac{11\pi}{40}
\end{aligned} \tag{7}$$

Jolany 2018 also gives, trivially:

$$\int_0^{\frac{\pi}{2}} \mathrm{d}t \sin^{2n}(t) = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2} \tag{8}$$

which is useful for taking a given integral and applying this formalism to attain a solution. For example:

$$\int_0^\infty \mathrm{d}x \frac{\sin^6(x)}{x^4} = \int_0^{\frac{\pi}{2}} \mathrm{d}t \sin^2(t) - \frac{2}{3}\int_0^{\frac{\pi}{2}} \mathrm{d}t \sin^4(t) = \frac{\pi}{4} - \frac{1}{4}\frac{\pi}{2} = \frac{\pi}{8} \tag{9}$$

Another creative ways to use Lobachevsky's Formula is:

$$\begin{aligned}
\int_0^\infty \mathrm{d}x \frac{\sin^2(x)}{x^4} \tan^2\left(\frac{x}{2}\right) &= \int_0^\infty \mathrm{d}x \frac{\sin^2(x)}{x^4} \left(\frac{1-\cos(x)}{\sin(x)}\right)^2 \\
&= \frac{1}{2}\int_0^\infty \mathrm{d}\left(\frac{x}{2}\right) \frac{\sin^4\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)^4} = \frac{1}{2}\frac{\pi}{3} = \frac{\pi}{6}
\end{aligned} \tag{10}$$

Another creative way to use Lobachevsky's Formula is:

$$\begin{aligned}
\int_{\frac{\pi}{2}}^\infty \frac{1}{x^2}\mathrm{dx} = \frac{2}{\pi} &= \int_0^\infty \mathrm{d}x \frac{\sin^2(x)}{x^2}\frac{1}{\sin^2(x)} - \int_0^{\frac{\pi}{2}} \mathrm{d}x \frac{1}{x^2} \\
&= \int_0^{\frac{\pi}{2}} \mathrm{d}x \left(\frac{1}{\sin^2(x)} - \frac{1}{x^2}\right) = \int_0^{\frac{\pi}{2}} \mathrm{d}x \sum_{k=1}^{\infty} \frac{1}{(x+k\pi)^2} + \frac{1}{(x-k\pi)^2} \\
&= \frac{\sum_{k=1}^{\infty}\left(\frac{1}{k-\frac{1}{2}} - \frac{1}{k+\frac{1}{2}}\right)}{\pi}
\end{aligned} \tag{11}$$

where the integral is expressed as a function of two terms over the opposite bounds to the initial expression, and the second and third lines are attained by Equation 4. Finally, in the next post we will use Lobachevsky's formula to find the Fourier Transform of various functions. This concludes this blog post.