Title: A Strange Observation of Fourier Series
Author: Josh Myers

June 30, 2025

This short post will highlight an interesting observation I've made recently while studying Fourier series. Consider the general function $f(x)$:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i \frac{n}{L} x} \tag{1}$$

where $\frac{-L}{2} < x < \frac{L}{2}$. As observed in my last post on truncation of infinite series, c_n is trivially:

$$c_n = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}x' f(x') \, e^{-2\pi i \frac{n}{L} x'} \tag{2}$$

Now substituting Equation 2 into Equation 1, then using the result from the "Finite Sums of Complex Numbers" blog post, we find the following:

$$\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}x' f(x') \, e^{2\pi i \frac{n}{L}(x-x')} \right) \\
&= \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}x' f(x') \sum_{n=-\infty}^{\infty} \left(e^{2\pi i \frac{n}{L}(x-x')} \right) \\
&= \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}x' f(x') \left(1 + 2 \sum_{n=1}^{\infty} \left(\cos\left(2\pi n \frac{(x-x')}{L} \right) \right) \right) \\
&= \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}x' f(x') \lim_{N\to\infty} \left(\sin\left(2\pi N \frac{(x-x')}{L} \right) \cot\left(\pi \frac{(x-x')}{L} \right) \right) \\
&= \lim_{N\to\infty} \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}x' f(x') \sin\left(2\pi N \frac{(x-x')}{L} \right) \cot\left(\pi \frac{(x-x')}{L} \right) \\
&= \lim_{N\to\infty} \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}x' f(x') \left[\frac{\sin\left(2\pi N \frac{(x-x')}{L} \right) \cot\left(\pi \frac{(x-x')}{L} \right)}{L} \right]
\end{aligned} \tag{3}$$

where in the square brackets we highlight a function that behaves similarly to the Dirac Delta function, $\delta(x-x')$. However, this function is not a Dirac Delta, since although it goes to ∞ as $x' \to x$, its value is not 0 everywhere else. Rather, the function has an infinite frequency periodic term, meaning that when integrating it with a "reasonable" function, we should attain an answer that is 0. After all, in any neighbourhood around $x' = x_o$, we see an infinite number of peaks and valleys contributed from the periodic term, meaning that the values from the multiplying functions get averaged to 0. However, when $x' \to x$, we get a diverging behaviour from the cotangent function, which leads to the infinite behaviour that is characteristic to the typical delta function. Furthermore, we get a function that integrates to 0 everywhere except where $x' \to x$ (where the function instead goes to infinity). Thus, it is unsurprising that we get the $f(x)$ as expected at the end of the integration. I have confirmed this observation numerically.

Now I will outline my plan for future posts and give some tidbits that I believe are useful for uncovering what I seek. I aim to establish a differential equation capable of discerning what $f(x)$ must be in order to lead to some Fourier series. That is, given some $c_n = g(n)$, what is the $f(x)$ needed to produce the $g(n)$ for all n? It is clear that knowing the integral of $f(x)$ multiplied by the Fourier complex exponential (*for all* n) is enough to determine $f(x)$ fully in the context of a Fourier series. Of course, the idea of knowing the infinite number of integrals is key, since it is clear that there are an infinite number of $f(x)$ that produce the desired $g(n_o)$ for a single $n = n_o$. However, this is the extent that I have gotten for this problem thus far. To complete this post, I will derive a well-known result: the Fourier Series of the Bernoulli polynomials of order n. This will be useful when using the Euler-Maclaurin formula on the general Fourier series in a future post. Bernoulli polynomials can be defined by the of $\frac{\mathrm{d}B_n}{\mathrm{d}x} = nB_{n-1}(x)$ (with $B_1(0) = -\frac{1}{2}$ and $B_1(1) = \frac{1}{2}$ and $B_0(x) = 1$). In general, $B_n(0) = B_n(1) = B_n$ for $n \geq 2$, where B_n is the Bernoulli number, defined via a variety of relations. Knowledge of the Bernoulli number and the differential relation $\frac{\mathrm{d}B_n}{\mathrm{d}x} = nB_{n-1}(x)$ is enough to fully define the Bernoulli polynomials. Now that we have established the relation, let's calculate the Fourier Series in question using iterative integration by parts:

$$\begin{aligned}
c_m &= \int_0^1 \mathrm{d}x B_n(x) e^{-2\pi imx} \\
&= \left(\frac{1}{-2\pi im} B_n(x) e^{-2\pi imx}\right)_{x=0}^{x=1} + \frac{1}{2\pi i}\frac{n}{m}\int_0^1 \mathrm{d}x B_{n-1}(x) e^{-2\pi imx} \\
&= \left(\frac{1}{2\pi i}\right)^{n-1} \frac{n!}{m^{n-1}} \int_0^1 \mathrm{d}x B_1(x) e^{-2\pi imx} \\
&= \left(\frac{1}{2\pi i}\right)^{n-1} \frac{n!}{m^{n-1}} \left(-\frac{1}{2\pi im}\left(B_1(1) - B_1(0)\right) + \frac{1}{2\pi im}\int_0^1 e^{-2\pi imx}\right) \\
&= -\frac{n!}{(2\pi i)^n}\frac{1}{m^n}
\end{aligned} \tag{4}$$

The Bernoulli Polynomials are thus:

$$B_n(x) = -\frac{n!}{(2\pi i)^n} \sum_{m=-\infty}^{\infty} \frac{e^{2\pi imx}}{m^n} \tag{5}$$

where the infinite sum does not include $m = 0$ (it vanishes, since $B_k(0) = B_k(1)$ for $k \geq 2$). This sum holds for $0 < x < 1$. The interesting possibility of $n \to \infty$ is:

$$\begin{aligned}
\lim_{n\to\infty} \frac{B_n(x)}{n!} &= -\frac{2}{(2\pi i)^n}\cos(2\pi x) \;\; \text{for } n \text{ even} \\
\lim_{n\to\infty} \frac{B_n(x)}{n!} &= -\frac{2i}{(2\pi i)^n}\sin(2\pi x) \;\; \text{for } n \text{ odd}
\end{aligned} \tag{6}$$

The Bernoulli polynomials goes to trigonometric terms as $n \to \infty$. This concludes this blog post.