Title: Fractional Calculus: Free Parameter Equation 2
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In the last post, I introduced the following equation that is independent of the free parameter α, where $\alpha \in \mathbb{R}$ and $0 < \alpha < 1$:

$$f(x) = \alpha \int_0^1 \mathrm{d}t \, (1-t)^{\alpha-1} \sum_{j=0}^{\infty} \frac{\binom{\alpha}{j}}{j!} f^{(j)}(xt)(xt)^j \tag{1}$$

In the last post, I demonstrated the implications of this equation for $f(x) = e^{-x^2}$. To restate the fascinating feature of Equation 1, the value of the right side is independent of α, though it would seem that α is featured in a non-trivial way that does not easily cancel. In this post, I aim to address three points:

- That for functions f that can be Maclaurin expanded, Equation 1 leads to a trivial cancellation of the α dependence on the right side, leading to the known α independence given on the left side.
- That for the function $f(x) = \arctan(x)$, there exists a series that yields the value of π for all of the applicable values of two free parameters α and x.
- That for the associated α integrated relation shown at the end of the last post, that for the Maclaurin expanded function $f(x)$, there exists a particularly interesting relation between a complicated integral and a much simpler one (which we will examine further in future posts).

To address the first point, we show the following:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n &= \sum_{j=0}^{\infty} \sum_{n=j}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \alpha \binom{\alpha}{j} \binom{n}{j} \int_0^1 \mathrm{d}t \, (1-t)^{\alpha-1} t^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \mathrm{B}(n+1, \alpha) \sum_{j=0}^{n} \alpha \binom{\alpha}{j} \binom{n}{j} \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \end{aligned} \tag{2}$$

where $\sum_{j=0}^{n} \alpha \binom{\alpha}{j} \binom{n}{j} = \frac{1}{\mathrm{B}(n+1,\alpha)}$. Thus, it is easy to see that for Maclaurin expanded functions, the Equation 2 leads to a trivial cancellation of the α dependence. Thus, the first purpose of this post is addressed. Now, for the second purpose, we get the following relation for $\arctan(x)$:

$$\arctan(x) = \frac{\alpha\sqrt{\pi}x}{2} \sum_{j=0}^{\infty} \frac{\binom{\alpha}{j}}{j!} 2^j \int_0^1 \mathrm{d}t \, (1-t)^{\alpha-1} t \tilde{F}_{32}\left(\frac{1}{2}, 1, 1; \frac{2-j}{2}, \frac{3-j}{2}; -t^2 x^2\right) \tag{3}$$

Carrying out the integral, we get:

$$\begin{aligned}\arctan(x) &= \frac{\alpha\pi x}{4}\sum_{j=0}^{\infty}\frac{\binom{\alpha}{j}}{j!}2^{j-\alpha}\Gamma(\alpha)\,\tilde{F}_{54}\left(\frac{1}{2},1,1,1,\frac{3}{2};\frac{2+\alpha}{2},\frac{3+\alpha}{2},\frac{2-j}{2},\frac{3-j}{2};-x^2\right)\\ \implies \pi &= \frac{\arctan(x)/x}{\frac{\alpha}{4}\sum_{j=0}^{\infty}\frac{\binom{\alpha}{j}}{j!}2^{j-\alpha}\Gamma(\alpha)\,\tilde{F}_{54}\left(\frac{1}{2},1,1,1,\frac{3}{2};\frac{2+\alpha}{2},\frac{3+\alpha}{2},\frac{2-j}{2},\frac{3-j}{2};-x^2\right)}\end{aligned} \tag{4}$$

Thus, the second purpose of this document has been addressed. Strangely, this formula seems to fail numerically for $x \to 1$ ($0 \le \alpha \le 1$). Now, we begin examining the third purpose, which we will examine more in future posts. Recall the α integration of Equation 1 found in the last post:

$$\int_0^1 \mathrm{d}t\left(\frac{\ln(1-t)+\frac{t}{1-t}}{\ln^2(1-t)}f(xt)+\sum_{j=1}^{\infty}\sum_{k=0}^{j-1}\frac{\left[{j \atop k+1}\right]}{(j!)^2}\frac{f^{(j)}(xt)\,\gamma(k+3,-\ln(1-t))}{(-1)^j(1-t)\ln^{k+3}(1-t)}(xt)^j\right)=f(x) \tag{5}$$

If we Maclaurin series expand $f(x)$ and equate powers of x^n, we get the following relation:

$$\int_0^1 \mathrm{d}t\left(\frac{\ln(1-t)+\frac{t}{1-t}}{\ln^2(1-t)}t^n+\sum_{j=1}^{n}\sum_{k=0}^{j-1}\frac{\gamma(k+3,-\ln(1-t))}{(1-t)\ln^{k+3}(1-t)}t^n\frac{\left[{j \atop k+1}\right]\binom{n}{j}}{(-1)^j j!}\right)=1 \tag{6}$$

To compute the first integral in Equation 6, we use the following anti-derivative identities:

$$\begin{aligned}\left(J^n\frac{1}{\ln(t)}\right)(x) &= \sum_{p=0}^{n-1}\left(\frac{(-1)^p x^{n-1-p}}{(n-1-p)!}\frac{\mathrm{Ei}((p+1)\ln(x))}{p!}\right)+C\\ \int_0^1 \mathrm{d}t\frac{\frac{t^{n+1}}{1-t}}{\ln^2(1-t)} &= -(n+1)\int_0^1 \mathrm{d}t\,\frac{t^n}{\ln(1-t)} = -(n+1)\int_0^1 \mathrm{d}t\,\frac{(1-t)^n}{\ln(t)}\end{aligned} \tag{7}$$

where here $\mathrm{Ei}(\ln(x)) = \mathrm{Li}(x) = \int_{-\infty}^{\ln(x)} \mathrm{d}t\,\frac{e^u}{u}$ are the exponential and logarithmic integrals, respectively, and the second line was obtained by integration by parts. The first identity can be used to find the solution to the first term in Equation 6:

$$\begin{aligned}\int_0^1 \mathrm{d}t\,\frac{(1-t)^n}{\ln(t)} &= \left[n!\sum_{q=0}^{n}\frac{(1-t)^{n-q}}{(n-q)!}\sum_{p=0}^{q}\left(\frac{t^{q-p}}{(q-p)!}\frac{\mathrm{Ei}((p+1)\ln(t))}{(-1)^p p!}\right)\right]_{t\to 0}^{t\to 1}\\ &= \lim_{t\to 1}\sum_{p=0}^{n}(-1)^p\binom{n}{p}\mathrm{Ei}((p+1)\ln(t))\end{aligned} \tag{8}$$

The second term in Equation 6 is found easily by making use of the second line of Equation 7 and Equation 8 result, which yields:

$$\int_0^1 \mathrm{d}t\frac{\frac{(1-t)^{n+1}}{t}}{\ln^2(t)} = -(n+1)\lim_{t\to 1}\sum_{p=0}^{n}(-1)^p\binom{n}{p}\mathrm{Ei}((p+1)\ln(t)) \tag{9}$$

Equation 8 and Equation 9 combine to give the interesting result:

$$\int_0^1 \mathrm{d}t\sum_{j=1}^{n}\sum_{k=0}^{j-1}\frac{\gamma(k+3,-\ln(1-t))}{(1-t)\ln^{k+3}(1-t)}t^n\frac{\left[{j \atop k+1}\right]\binom{n}{j}}{(-1)^j j!} = 1+n\lim_{t\to 1}\sum_{p=0}^{n}(-1)^p\binom{n}{p}\mathrm{Ei}((p+1)\ln(t)) \tag{10}$$

This concludes this blog post.