Title: Observations of Logarithmic Integrals 2
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In the previous post, I showed the value of the following integral:

$$I_1(x) = \int_0^x \mathrm{d}t \, \ln\left(\frac{x-t}{1-t}\right) \ln^2(1-t) \tag{1}$$

using the following identity:

$$\lim_{r\to 0}\left(\frac{z^r - 1}{r}\right) = \ln(z) \tag{2}$$

In this post, using the same identity, we will find the value of the following integral:

$$I_1(x) = \int_0^x \mathrm{d}t \, \ln^2(t) \ln^2(1-t) \tag{3}$$

Through using Equation 2 four times, carrying out the integration in t on each term, then evaluating the limits, we get the following result:

$$\begin{aligned}
I_1(x) &= 18 - \frac{2\pi^2}{3} - \frac{\pi^4}{15} + \frac{1}{18}\left(\pi^2 - 6\right)^2 - 2(1-x)\ln^2(1-x) - 2x\ln^2(1-x)\ln(x) \\
&+ x\ln^2(1-x)\ln^2(x) + 2\psi_2(2) - 4\zeta(3) - 2\gamma(1-x)\, F_{3,2}^{(0,0,2;0,0;0)}(1,1,0;2,2;1-x) \\
&+ (1-x)\left(2\ln(1-x)\, F_{3,2}^{(0,0,2;0,0;0)}(1,1,0;2,2;1-x) + \tilde{F}_{3,2}^{(0,0,2;0,1;0)}(1,1,0;2,2;1-x)\right) \\
&+ (1-x)\left(\tilde{F}_{3,2}^{(0,0,2;1,0;0)}(1,1,0;2,2;1-x) + \tilde{F}_{3,2}^{(0,1,2;0,0;0)}(1,1,0;2,2;1-x)\right) \\
&+ (1-x)\, \tilde{F}_{3,2}^{(1,0,2;0,0;0)}(1,1,0;2,2;1-x)
\end{aligned} \tag{4}$$

where $\psi_n(x)$ is the polygamma function, $\zeta(s)$ is the Riemann Zeta function, F_{32} is the hypergeometric function and $\tilde{F}_{32}$ is the regularized hypergeometric function. Mathematica and Wolfram Alpha both tap out if asked to solve the Equation 3 directly. However, with the help of the algorithm described in the last post, this result was found - I have checked this numerically!

However, there are many logarithmic integrals I have still not succeeded in solving, such as:

$$\begin{aligned}
I_2(x) &= \int_0^x \mathrm{d}t \, \ln(1-t)\ln(t)\ln(1+t) \\
I_3(x) &= \int_0^x \mathrm{d}t \, \mathrm{Li}_2(t)\, \mathrm{Li}_2(1-t)
\end{aligned} \tag{5}$$

Wolfram Alpha gives a solution for I_2 that is tremendously long (certainly too long to write here) - the algorithm that I employed to attain I_1 does not seem to work for I_2 for some reason. For

I_3, matters are complicated by presence of the polylogarithm function (rather than simply the logarithm). However, I do have some hopes of solving these one in future posts. It is noteworthy that for $x = 1$, I_3 is the following, which is solveable through expanding both polylogarithms in Equation 5, integrating to achieve the complete beta function, then using Mathematica to solve the double sum using the identity $B\left(p,q\right) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$:

$$I_3\left(1\right) = 6 + \frac{\pi^2}{120}\left(\pi^2 - 80\right) \tag{6}$$

The difficulty with solving I_3 for the general x is that to get the Li_2 in terms of the logarithm, more integrals must be implemented, as can be seen in the Equation below:

$$\mathrm{Li}_2\left(t\right) = -\int_0^t \mathrm{d}z \frac{\ln\left(1-z\right)}{z} \tag{7}$$

Thus, more integral evaluations must be carried out, and in order to evaluate the integral in terms of t, the order of integration must be interchanged, as t is a bound on the z integral. This is not especially a problem on its own (integration order can be switched), but it leads to enough integrals that I have not been able to find the closed form solution for $I_3\left(x\right)$ as of yet. This concludes this blog post.