Title: Re-examination of Stieltjes Constants
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From the last post on the Stieltjes Constants, I ended by arriving at the last expression:

$$\gamma_n = \lim_{m\to\infty} -n! \prod_{\mu=0}^{n} \sum_{w_\mu=\delta_{\mu 0}}^{\infty} \frac{\left(1-\frac{1}{m}\right)^{n+\sum_{d=0}^{n} w_d}}{\mu+\sum_{d=0}^{\mu} w_d} + \int_0^1 \mathrm{d}t$$
$$\sum_{\omega=0}^{n} \left(\binom{n}{\omega} \frac{1-t^m}{1-t} \left(-\ln\left(-\ln\left(t\right)\right)\right)^{n-\omega} B_\omega \left(\left(-1\right)^0 \psi_0\left(1\right), ..., \left(-1\right)^{n-1} \psi_{n-1}\left(1\right) \right) \right) \tag{1}$$

Series Expanding $\frac{1}{1-t}$ as the geometric series, we get:

$$\gamma_n = \lim_{m\to\infty} -n! \prod_{\mu=0}^{n} \sum_{w_\mu=\delta_{\mu 0}}^{\infty} \frac{\left(1-\frac{1}{m}\right)^{n+\sum_{d=0}^{n} w_d}}{\mu+\sum_{d=0}^{\mu} w_d} + \int_0^1 \mathrm{d}t$$
$$\sum_{\ell=0}^{\infty} \sum_{\omega=0}^{n} \left(\binom{n}{\omega} \left(t^\ell - t^{\ell+m}\right) \left(-\ln\left(-\ln\left(t\right)\right)\right)^{n-\omega} B_\omega \left(\left(-1\right)^0 \psi_0\left(1\right), ..., \left(-1\right)^{n-1} \psi_{n-1}\left(1\right) \right) \right) \tag{2}$$

Substituting $t = e^{-u}$, $\mathrm{d}t = -e^{-u}\mathrm{d}u$, we get:

$$\gamma_n = \lim_{m\to\infty} -n! \prod_{\mu=0}^{n} \sum_{w_\mu=\delta_{\mu 0}}^{\infty} \frac{\left(1-\frac{1}{m}\right)^{n+\sum_{d=0}^{n} w_d}}{\mu+\sum_{d=0}^{\mu} w_d}$$
$$+ \int_0^\infty \mathrm{d}u \sum_{\ell=0}^{\infty} \sum_{\omega=0}^{n} \left(\binom{n}{\omega} \left(e^{-(\ell+1)u} - e^{-(\ell+m+1)u}\right) \left(-\ln\left(u\right)\right)^{n-\omega} B_\omega\left(\Psi_n\right) \right) \tag{3}$$

Now, we make a substitution to bring the integral in the second line closer to a derivative of the gamma function.

$$
\begin{aligned}
\gamma_n = &\lim_{m\to\infty} -n! \prod_{\mu=0}^{n} \sum_{w_\mu=\delta_{\mu 0}}^{\infty} \frac{\left(1-\frac{1}{m}\right)^{n+\sum_{d=0}^{n} w_d}}{\mu+\sum_{d=0}^{\mu} w_d} \\
&+ \sum_{\ell=0}^{\infty}\sum_{\omega=0}^{n}\left(\binom{n}{\omega}\frac{(-1)^{n-\omega}}{\ell+1}\int_0^\infty \mathrm{d}u\, e^{-u}\ln^{n-\omega}\left(\frac{u}{\ell+1}\right)B_\omega\left(\Psi_n\right)\right) \\
&- \sum_{\ell=0}^{\infty}\sum_{\omega=0}^{n}\left(\binom{n}{\omega}\frac{(-1)^{n-\omega}}{\ell+m+1}\int_0^\infty \mathrm{d}u\, e^{-u}\ln^{n-\omega}\left(\frac{u}{\ell+m+1}\right)B_\omega\left(\Psi_n\right)\right) \\
= &\lim_{m\to\infty} -n! \prod_{\mu=0}^{n} \sum_{w_\mu=\delta_{\mu 0}}^{\infty} \frac{\left(1-\frac{1}{m}\right)^{n+\sum_{d=0}^{n} w_d}}{\mu+\sum_{d=0}^{\mu} w_d} \\
&+ \sum_{\ell=0}^{\infty}\sum_{\omega=0}^{n}\sum_{\sigma=0}^{n-\omega}\binom{n-\omega}{\sigma}\binom{n}{\omega}\ln^{n-\omega-\sigma}(\ell+1)\,B_\omega\left(\Psi_n\right)\frac{(-1)^\sigma}{\ell+1}\int_0^\infty \mathrm{d}u\, e^{-u}\ln^\sigma(u) \\
&- \sum_{\ell=0}^{\infty}\sum_{\omega=0}^{n}\sum_{\sigma=0}^{n-\omega}\binom{n-\omega}{\sigma}\binom{n}{\omega}\frac{\ln^{n-\omega-\sigma}(\ell+m+1)\,B_\omega\left(\Psi_n\right)(-1)^\sigma}{\ell+m+1}\int_0^\infty \mathrm{d}u\, e^{-u}\ln^\sigma(u) \\
= &\lim_{m\to\infty} -n! \prod_{\mu=0}^{n} \sum_{w_\mu=\delta_{\mu 0}}^{\infty} \frac{\left(1-\frac{1}{m}\right)^{n+\sum_{d=0}^{n} w_d}}{\mu+\sum_{d=0}^{\mu} w_d} \\
&+ \sum_{\ell=0}^{\infty}\sum_{\omega=0}^{n}\sum_{\sigma=0}^{n-\omega}\binom{n-\omega}{\sigma}\binom{n}{\omega}\ln^{n-\omega-\sigma}(\ell+1)\,B_\omega\left(\Psi_n\right)\frac{(-1)^\sigma}{\ell+1}\Gamma^{(\sigma)}(1) \\
&- \sum_{\ell=0}^{\infty}\sum_{\omega=0}^{n}\sum_{\sigma=0}^{n-\omega}\binom{n-\omega}{\sigma}\binom{n}{\omega}\frac{\ln^{n-\omega-\sigma}(\ell+m+1)\,B_\omega\left(\Psi_n\right)(-1)^\sigma}{\ell+m+1}\Gamma^{(\sigma)}(1)
\end{aligned} \tag{4}
$$

Where if we use similar reasoning as in the last post:

$$
\Gamma^{(\sigma)}(1) = B_\sigma\left(\psi_0(1), \psi_1(1), ..., \psi_{n-1}(1)\right)
$$

Then we get an expression in terms of the complete exponential Bell polynomials:

$$
\gamma_n = \lim_{m\to\infty} -\frac{\ln^{n+1}(m)}{n+1} + \sum_{\omega=0}^{n}\sum_{\sigma=0}^{n-\omega}\binom{n-\omega}{\sigma}\binom{n}{\omega}B_\omega\left(\Psi_n^-\right)(-1)^\sigma B_\sigma\left(\Psi_n\right)\sum_{\ell=1}^{m}\left(\frac{\ln^{n-\omega-\sigma}(\ell)}{\ell}\right) \tag{5}
$$

which I have confirmed numerically for $n = 1$ and $n = 2$. Though I have to consider this further conceptually, there surprisingly isn't a simplification to the similar looking definition of the Stieltjes Constants:

$$
\gamma_n = \lim_{m\to\infty}\sum_{k=1}^{\infty}\frac{\ln^n(k)}{k} - \frac{\ln^{n+1}(m)}{n+1}
$$

Overall, an interesting implicit relation exists for the Stieltjes Constants:

$$
\begin{aligned}
\gamma_n &= \lim_{m\to\infty} -\frac{\ln^{n+1}(m)}{n+1} + \sum_{\omega=0}^{n}\sum_{\sigma=0}^{n-\omega}\binom{n-\omega}{\sigma}\binom{n}{\omega}B_\omega\left(\Psi_n^-\right)(-1)^\sigma B_\sigma\left(\Psi_n\right)\left(\gamma_{n-\omega-\sigma} + \frac{\ln^{n-\omega-\sigma+1}(m)}{n-\omega-\sigma+1}\right) \\
0 &= \lim_{m\to\infty}\sum_{\omega=0}^{n}\sum_{\sigma=0}^{n-\omega}\left(1-\delta_{\sigma 0}\delta_{\omega 0}\right)\binom{n-\omega}{\sigma}\binom{n}{\omega}B_\omega\left(\Psi_n^-\right)(-1)^\sigma B_\sigma\left(\Psi_n\right)\left(\gamma_{n-\omega-\sigma} + \frac{\ln^{n-\omega-\sigma+1}(m)}{n-\omega-\sigma+1}\right)
\end{aligned} \tag{6}
$$

where:

$$\sum_{\omega=0}^{n}\sum_{\sigma=0}^{n-\omega}\binom{n-\omega}{\sigma}\binom{n}{\omega}B_{\omega}\left(\Psi_{n}^{-}\right)(-1)^{\sigma}B_{\sigma}\left(\Psi_{n}\right)=1$$

which holds for all n, and where:

$$\Psi_{n}=\left\{\psi_{0}\left(1\right),\psi_{1}\left(1\right),...,\psi_{n-1}\left(1\right)\right\}$$

$$\Psi_{n}^{-}=\left\{(-1)^{0}\psi_{0}\left(1\right),-\psi_{1}\left(1\right),...,(-1)^{n-1}\psi_{n-1}\left(1\right)\right\}$$

Note the interesting relation between the complete exponential Bell Polynomials and the Stieltjes Constants in Equation 6. This relation is a trivial simplification:

$$\lim_{m\to\infty}\sum_{\omega=0}^{n}\sum_{\sigma=0}^{n-\omega}\left(1-\delta_{\sigma0}\delta_{\omega0}\right)\binom{n-\omega}{\sigma}\binom{n}{\omega}B_{\omega}\left(\Psi_{n}^{-}\right)(-1)^{\sigma}B_{\sigma}\left(\Psi_{n}\right)\left(\gamma_{n-\omega-\sigma}+\frac{\ln^{n-\omega-\sigma+1}(m)}{n-\omega-\sigma+1}\right)=0$$

This property is a result of the Bell Polynomials - in generality:

$$\sum_{\omega=0}^{n}\sum_{\sigma=0}^{n-\omega}\left(1-\delta_{\sigma0}\delta_{\omega0}\right)\binom{n-\omega}{\sigma}\binom{n}{\omega}B_{\omega}\left(\Phi_{n}^{-}\right)(-1)^{\sigma}B_{\sigma}\left(\Phi_{n}\right)f\left(n-\omega-\sigma\right)=0 \tag{7}$$

where:

$$\Phi_{n}=\left\{x_{0},x_{1},...,x_{n-1}\right\}$$

$$\Phi_{n}^{-}=\left\{(-1)^{0}x_{0},(-1)^{1}x_{1},...,(-1)^{n-1}x_{n-1}\right\}$$

where x_m represents a general sequence of numbers, and $f\left(n-\omega-\sigma\right)$ is a general function with this integer argument input. The recovery of true relations based on implementing the standard definition of Stieltjes Constants serves as excellent verification of the correctness of the investigation carried here in this post and the last. However, before ending this post, I'd like to reconnect these ideas to the Riemann Zeta function. So consider the following calculation:

$$\begin{aligned}
\zeta\left(s\right)+\frac{1}{1-s}&=\sum_{n=0}^{\infty}\frac{\gamma_{n}}{n!}\left(1-s\right)^{n}\\
&=\sum_{n=0}^{\infty}\sum_{\omega=0}^{n}\sum_{\sigma=0}^{n-\omega}\binom{n-\omega}{\sigma}\binom{n}{\omega}B_{\omega}\left(\mathbf{1}_{n}^{-}\right)(-1)^{\sigma}B_{\sigma}\left(\mathbf{1}_{n}\right)\frac{\gamma_{n-\omega-\sigma}}{n!}\left(1-s\right)^{n}\\
&=\sum_{\omega=0}^{\infty}\sum_{n=0}^{\infty}\sum_{\sigma=0}^{n}\binom{n}{\sigma}\binom{n+\omega}{\omega}B_{\omega}\left(\mathbf{1}_{n+\omega}^{-}\right)(-1)^{\sigma}B_{\sigma}\left(\mathbf{1}_{n+\omega}\right)\frac{\gamma_{n-\sigma}}{(n+\omega)!}\left(1-s\right)^{n+\omega}\\
&=\sum_{\omega=0}^{\infty}\sum_{\sigma=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{\sigma!n!\omega!}B_{\omega}\left(\mathbf{1}_{n+\omega+\sigma}^{-}\right)(-1)^{\sigma}B_{\sigma}\left(\mathbf{1}_{n+\omega+\sigma}\right)\gamma_{n}\left(1-s\right)^{n+\omega+\sigma}\\
&=\sum_{n=0}^{\infty}\frac{\gamma_{n}}{n!}\left(1-s\right)^{n}\sum_{\omega=0}^{\infty}B_{\omega}\left(\mathbf{1}_{n+\omega+\sigma}^{-}\right)\frac{\left(1-s\right)^{\omega}}{\omega!}\sum_{\sigma=0}^{\infty}B_{\sigma}\left(\mathbf{1}_{n+\omega+\sigma}\right)\frac{\left(s-1\right)^{\sigma}}{\sigma!}
\end{aligned} \tag{8}$$

where by a defining relation of the complete exponential Bell Polynomials:

$$e^{\sum_{n=1}^{\infty}x_{n}\frac{t^{n}}{n!}}=\sum_{n=0}^{\infty}B_{n}\left(x_{1},x_{2},...,x_{n}\right)\frac{t^{n}}{n!}$$

and:

$$B_\sigma\left(\mathbf{X}_{n+\omega+\sigma}\right)=B_\sigma\left(\mathbf{X}_\sigma\right)$$

This means that:

$$B_\sigma\left(\mathbf{1}_{n+\omega+\sigma}\right)=B_\sigma\left(\mathbf{1}_\sigma\right), B_\omega\left(\mathbf{1}^-_{n+\omega+\sigma}\right)=B_\omega\left(\mathbf{1}^-_\omega\right)$$

and:

$$\sum_{\sigma=0}^{\infty} B_\sigma\left(\mathbf{1}_\sigma\right)\frac{(1-s)^\sigma}{\sigma!}=e^{\sum_{\sigma=1}^{\infty}\frac{(1-s)^\sigma}{\sigma!}}=e^{e^{1-s}-1}$$

$$\sum_{\omega=0}^{\infty} B_\omega\left(\mathbf{1}^-_\omega\right)\frac{(s-1)^\omega}{\omega!}=e^{\sum_{\omega=1}^{\infty}(-1)^{\omega-1}\frac{(s-1)^\omega}{\omega!}}=e^{-\sum_{\omega=1}^{\infty}\frac{(1-s)^\omega}{\omega!}}=e^{-e^{1-s}+1}$$

Now, we can make progress on Equation 8:

$$\begin{aligned}\zeta(s)+\frac{1}{1-s}&=\sum_{n=0}^{\infty}\frac{\gamma_n}{n!}(1-s)^n\sum_{\omega=0}^{\infty}B_\omega\left(\mathbf{1}^-_\omega\right)\frac{(1-s)^\omega}{\omega!}\sum_{\sigma=0}^{\infty}B_\sigma\left(\mathbf{1}_\sigma\right)\frac{(s-1)^\sigma}{\sigma!}\\ 1&=e^{e^{1-s}-1}e^{-e^{1-s}+1}=1\end{aligned}\tag{9}$$

as expected - a strong symbolic (critically, non-numeric) confirmation of the calculations herein. This concludes this blog post.