Title: Variations on the Euler-Maclaurin Formula
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July 1, 2025

In this post, I will discuss a variation on a simple order 2 Euler-Maclaurin Formula:

$$\sum_{y=1}^{N} f(y) = \int_1^N \mathrm{d}x f(x) + \frac{f(N)+f(1)}{2} + \frac{1}{12}\left(f'(N) - f'(1)\right) - \frac{1}{2}\int_1^N \mathrm{d}x_0\, P_2(x_0)\, f''(x_0) \tag{1}$$

where $P_2(x_0)$ is the second periodic Bernoulli Polynomial, which is defined as follows:

$$P_2(x_0) = B_2(x_0 - \lfloor x_0 \rfloor) \tag{2}$$

where $B_2(x_0) = x_0^2 - x_0 + \frac{1}{6}$ is the second Bernoulli polynomial. From this, it seems from Equation 1 that we have traded a sum in f for a sum of an integral involving f. From Wikipedia, it is stated that for R_p as below (Equation 1 is for $p = 2$), there is a useful bounding function:

$$\begin{aligned} R_p &= (-1)^{p+1} \int_1^N \mathrm{d}x_0 \frac{P_p(x_0)}{p!} f^{(p)}(x_0) \\ \mid R_p \mid &< \frac{2\zeta(p)}{(2\pi)^p} \int_1^N \mathrm{d}x_0 \mid f^{(p)}(x_0) \mid \end{aligned} \tag{3}$$

For algorithmic simplicity, let $F_0''(x_0) = -\frac{1}{2} f''(x_0)$. Though the integral in the second line of Equation 3 cannot be nicely done by hand (except in special cases where all zeroes of the pth derivative of f are known), it can be done computationally for many functions to help give an idea of the order of magnitude of the remainder associated with R_p when carrying out Equation 1. Note that $\zeta(p)$ in Equation 3 is the Riemann Zeta function.

It is curious to consider the benefit of applying the Euler-Maclaurin formula *iteratively* for the following sum which can be expressed as R_2:

$$R_2 = \sum_{q_0=1}^{N-1} \int_{q_0}^{q_0+1} \mathrm{d}x_0\, B_2(x_0 - q_0)\, F_0''(x_0) \tag{4}$$

such that Equation 1 is applied for the sum from $q_0 \in \mathbb{W}$ between 1 and $N-1$ within R_2. This application would lead to a remainder term $R_2^{(1)}$ *within* R_2.

$$\begin{aligned} R_2 = &\int_1^{N-1} \mathrm{d}x_1 \int_{x_1}^{x_1+1} \mathrm{d}x_0 B_2(x_0 - x_1)\, F_0''(x_0) + \frac{1}{2}\int_{N-1}^{N} \mathrm{d}x_0 B_2(x_0 - (N-1))\, F_0''(x_0) \\ &+ \frac{1}{2}\int_1^2 \mathrm{d}x_0 B_2(x_0 - 1)\, F_0''(x_0) + \frac{1}{72}\left(\left(F_0''(N) - F_0''(N-1)\right) - \left(F_0''(2) - F_0''(1)\right)\right) \\ &- \frac{1}{6}\left(\int_{N-1}^{N} \mathrm{d}x_0 B_1(x_0 - (N-1))\, F_0''(x_0) - \int_1^2 \mathrm{d}x_0 B_1(x_0 - 1)\, F_0''(x_0)\right) + R_2^{(1)} \end{aligned} \tag{5}$$

where $R_2^{(1)}$ is:

$$R_2^{(1)} = \int_1^{N-1} \mathrm{d}x_1 P_2(x_1) F_1''(x_1) = \sum_{q_1=1}^{N-2} \int_{q_1}^{q_1+1} \mathrm{d}x_1 B_2(x_1 - q_1) F_1''(x_1) \tag{6}$$

where:

$$F_1''(x_1) = -\frac{1}{12}\left(F_0'''(x_1+1) - F_0'''(x_1)\right) + \frac{1}{2}\left(F_0''(x_1+1) + F_0''(x_1)\right) - \left(F_0'(x_1+1) - F_0'(x_1)\right) \tag{7}$$

where F_1'' comes from taking two derivatives with respect to x_1 of $\int_{x_1}^{x_1+1} \mathrm{d}x_0\, B_2(x_0 - x_1) F_0''(x_0)$ (using Leibniz Integral Rule and $\frac{\partial B_n(x-x_2)}{\partial x_2} = -nB_{n-1}(x - x_2)$). These formula can be extended through generalizing a function:

$$F_k''(x_k) = -\frac{1}{2}\sum_{w=0}^{2}(-2)_{2-w}\left(B_w(1) F_{k-1}^{(w+1)}(x_k+1) - B_w(0) F_{k-1}^{(w+1)}(x_k)\right) \tag{8}$$

This generalized function $F_k''(x_k)$ can be used to rewrite the sum of $R_2^{(z)}$ by:

$$\begin{aligned}
\sum_{y=1}^{N} f(y) = &\int_1^N \mathrm{d}x f(x) + \frac{f(N)+f(1)}{2} + \frac{1}{12}\left(f'(N) - f'(1)\right) \\
&+ \sum_{k=0}^{N-3}\int_1^{N-k-1}\mathrm{d}x_{k+1}\int_{x_{k+1}}^{x_{k+1}+1}\mathrm{d}x_k B_2(x_k - x_{k+1}) F_k''(x_k) \\
&+ \frac{1}{2}\int_{N-k-1}^{N-k}\mathrm{d}x_k B_2(x_k - (N-k-1)) F_k''(x_k) + \frac{1}{2}\int_1^2 \mathrm{d}x_k B_2(x_k - 1) F_k''(x_k) \\
&+ \frac{1}{72}\left(\left(F_k''(N-k) - F_k''(N-k-1)\right) - \left(F_k''(2) - F_k''(1)\right)\right) \\
&- \frac{1}{6}\left(\int_{N-k-1}^{N-k}\mathrm{d}x_k B_1(x_k - (N-k-1)) F_k''(x_k) - \int_1^2 \mathrm{d}x_k B_1(x_k - 1) F_k''(x_k)\right) \\
&+ \int_1^2 \mathrm{d}x_{N-2} B_2(x_{N-2} - 1) F_{N-2}''(x_{N-2})
\end{aligned} \tag{9}$$

This elaborate formula (arrived at through relatively simple means) allows for us to begin to evaluate sums in a different way. It is perhaps too much to wish that we may find any sum easier through employing this formula. However, I find the following interesting for future investigation:

- Use this formula to attempt to evaluate a finite sum. This may provide interesting results to consider.
- Consider the idea of an infinite sum, where $N \to \infty$. Once such a sum has been evaluated, we can then consider an intractable one.
- Analyse the outcome of the Equation 8 for various starting $f(y)$, considering in particular the evolution of the inequality in Equation 3 as k progressively increases.

It will be interesting to consider these ideas in future posts. I encourage to reader to do out the calculations that I have carried out above themselves to understand the algorithmic definition of F_k'' that I recommend. This concludes this blog post.