Title: Complex Fourier Series
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Recall the Fourier Series definition discussed in the last post:

$$g(x) = c_0 \sum_{n\neq 0}^{(-\infty,\infty)} \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}t\, g(t)\, e^{2\pi i n \frac{x-t}{L}} \tag{1}$$

Now, define $h(x) = xg(x)$. Then integrate Equation 1 by parts:

$$\begin{aligned} g(x) = c_0 \, + & \sum_{n\neq 0}^{(-\infty,\infty)} e^{2\pi i n \frac{x}{L}} g\left(\frac{L}{2}\right) \frac{\mathrm{Ei}(-\pi i n)}{2} + g\left(-\frac{L}{2}\right) \frac{\mathrm{Ei}(\pi i n)}{2} \\ & - \sum_{n\neq 0}^{(-\infty,\infty)} \frac{e^{2\pi i n \frac{x}{L}}}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}t\, h'(t)\, \mathrm{Ei}\left(-2\pi i n \frac{t}{L}\right) \end{aligned} \tag{2}$$

It is true that a general real-valued function $g(x) = g_e(x) + g_o(x)$ can be written as a sum of even and odd functions. As well, a purely imaginary argument for the exponential integral can be written as $\mathrm{Ei}(iz) = \mathrm{ci}(z) + i\left(\mathrm{si}(z) + \frac{\pi}{2}\right)$, where $\mathrm{ci}(z) = -\int_z^\infty \mathrm{d}t \frac{\cos(t)}{t}$ and $\mathrm{si}(z) = -\int_z^\infty \mathrm{d}t \frac{\sin(t)}{t}$ are the cosine and sine integrals, respectively. With these specifications, we get:

$$\begin{aligned} g(x) = c_0 + & \sum_{n\neq 0}^{(-\infty,\infty)} e^{2\pi i n \frac{x}{L}} g_e\left(\frac{L}{2}\right) \mathrm{ci}(\pi n) - i e^{2\pi i n \frac{x}{L}} g_o\left(\frac{L}{2}\right) \left(\mathrm{si}(\pi n) + \frac{\pi}{2}\right) \\ & - \sum_{n\neq 0}^{(-\infty,\infty)} e^{2\pi i n \frac{x}{L}} \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}t\, h'(t) \left(\mathrm{ci}\left(2\pi n \frac{t}{L}\right) + i\left(-\mathrm{si}\left(2\pi n \frac{t}{L}\right) + \frac{\pi}{2}\right)\right) \end{aligned} \tag{3}$$

Now, the cosine and sine integrals have the following series expansions:

$$\begin{aligned} \mathrm{ci}(z) &= \gamma + \ln(z) + \sum_{k=1}^{\infty} \frac{\left(-z^2\right)^k}{2k\,(2k)!} \\ \mathrm{si}(z) &= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{z^{2k-1}}{(2k-1)\,(2k-1)!} \end{aligned} \tag{4}$$

where γ is the Euler-Mascheroni constant. Now, splitting the integral term into even and odd functions of h', we get:

$$
\begin{aligned}
g(x) = c_0 + \sum_{n\neq 0}^{(-\infty,\infty)} e^{2\pi i n \frac{x}{L}} g_e\left(\frac{L}{2}\right)\mathrm{ci}(\pi n) - i e^{2\pi i n \frac{x}{L}} g_o\left(\frac{L}{2}\right)\left(\mathrm{si}(\pi n) + \frac{\pi}{2}\right) \\
- \sum_{n\neq 0}^{(-\infty,\infty)} \frac{e^{2\pi i n \frac{x}{L}}}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}t \, \left(g_e(t) + t g_e'(t)\right)\left(\mathrm{ci}\left(2\pi n \frac{t}{L}\right) + i\frac{\pi}{2}\right) \\
+ \sum_{n\neq 0}^{(-\infty,\infty)} \frac{e^{2\pi i n \frac{x}{L}}}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}t \, \left(g_o(t) + t g_o'(t)\right)\left(i\,\mathrm{si}\left(2\pi n \frac{t}{L}\right) - \ln\left(2\pi n \frac{t}{L}\right)\right)
\end{aligned}
\tag{5}
$$

An important point is that $\ln(-x) = \ln(x) + i\pi$, making it even in real value but producing an additional imaginary value when $x < 0$. This makes the following calculation noteworthy:

$$
\begin{aligned}
-\frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}t \, \left(g_o(t) + t g_o'(t)\right)\ln\left(2\pi n \frac{t}{L}\right) &= -\frac{i\pi}{L}\int_{-\frac{L}{2}}^{0} \mathrm{d}t \, \left(g_o(t) + t g_o'(t)\right) \\
&= -\frac{i\pi}{2} g_o\left(-\frac{L}{2}\right) \\
&= \frac{i\pi}{2} g_o\left(\frac{L}{2}\right)
\end{aligned}
\tag{6}
$$

which cancels with the second term in the top line of Equation 5.

Now, we can calculate for some particular functions of $g(x)$. Let $g(x) = x$, and thus $c_0 = 0$, then we get:

$$
\begin{aligned}
g(x) &= -i \sum_{n\neq 0}^{(-\infty,\infty)} e^{2\pi i n \frac{x}{L}} \frac{L}{2}\mathrm{si}(\pi n) + \frac{e^{2\pi i n \frac{x}{L}}}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}t \, 2ti \, \mathrm{si}\left(2\pi n \frac{t}{L}\right) \\
&= -i \sum_{n\neq 0}^{(-\infty,\infty)} e^{2\pi i n \frac{x}{L}} \frac{L}{2}\mathrm{si}(\pi n) + i e^{2\pi i n \frac{x}{L}} \frac{L}{2}\left(\frac{(-1)^n}{n\pi} + \mathrm{si}(\pi n)\right) \\
&= \sum_{n\neq 0}^{(-\infty,\infty)} (-1)^n e^{2\pi i n \frac{x}{L}} \frac{iL}{2n\pi} \\
&= x
\end{aligned}
\tag{7}
$$

where the last line is derived from the appearance of the Fourier coefficient $c_n = (-1)^n \frac{iL}{2n\pi}$ which is characteristic of $g(x) = x$. So we get the expected result. Now, if we calculate the same for the particular value $g(x) = 1$, where $c_0 = 1$, we get:

$$
\begin{aligned}
g(x) &= 1 + \sum_{n\neq 0}^{(-\infty,\infty)} e^{2\pi i n \frac{x}{L}} \mathrm{ci}(\pi n) - \frac{e^{2\pi i n \frac{x}{L}}}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}t \, \left(\mathrm{ci}\left(2\pi n \frac{t}{L}\right) + i\frac{\pi}{2}\right) \\
&= 1 + \sum_{n\neq 0}^{(-\infty,\infty)} e^{2\pi i n \frac{x}{L}} \mathrm{ci}(\pi n) - e^{2\pi i n \frac{x}{L}}\left(\mathrm{ci}(\pi n) - \frac{\sin(\pi n)}{\pi n}\right) \\
&= 1
\end{aligned}
\tag{8}
$$

as expected. This concludes this blog post.