

**Title:** Fourier Series and the Bernoulli Polynomials  
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Previously, I have posted about the Fourier Series of an analytic function  $g(x)$ . To summarize these results, I'll re-derive the main results here:

$$\begin{aligned}
g(x) &= \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n \frac{x}{L}} = \sum_{m=0}^{\infty} \frac{g^{(m)}(0)}{m!} x^m \\
c_n &= \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx' g(x') e^{-2\pi i n \frac{x'}{L}} \\
&= \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx' e^{-2\pi i n \frac{x'}{L}} \sum_{m=0}^{\infty} \frac{g^{(m)}(0)}{m!} (x')^m \\
&= \frac{1}{L} \sum_{m=0}^{\infty} \frac{g^{(m)}(0)}{m!} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx' (x')^m e^{-2\pi i n \frac{x'}{L}} \\
&= \frac{1}{L} \sum_{m=0}^{\infty} \frac{g^{(m)}(0)}{m!} \left(\frac{L}{2\pi i n}\right)^{m+1} \int_{-\pi i n}^{\pi i n} dx' (x')^m e^{-x'} \\
&= \sum_{m=0}^{\infty} \frac{g^{(m)}(0)}{m!} \left(\frac{L}{2\pi i n}\right)^m \frac{\gamma(m+1, \pi i n) - \gamma(m+1, -\pi i n)}{2\pi i n}
\end{aligned} \tag{1}$$

Now, I am going to show you a method that *doesn't* work, by using the identity for the lower incomplete gamma function (only valid for  $s$  not an integer):

$$\gamma(s, z) = \int_0^z dt t^{s-1} e^{-t} = z^s e^{-z} \sum_{k=0}^{\infty} \frac{z^k}{(s)_{k+1}}$$

we obtain the following double infinite series result:

$$\begin{aligned}
c_n &= \sum_{m=0}^{\infty} \frac{g^{(m)}(0)}{m!} \left(\frac{L}{2\pi i n}\right)^m \sum_{k=0}^{\infty} (-1)^k \left(\frac{(\pi i n)^{k+m+1} - (-\pi i n)^{k+m+1}}{2\pi i n (m+1)_{k+1}}\right) \\
&= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \left(\frac{L}{2}\right)^m \frac{g^{(m)}(0)}{(m+k+1)!} (\pi i n)^k (-1)^k \left(\frac{1 - (-1)^{m+k+1}}{2}\right)
\end{aligned} \tag{2}$$

Now we set  $x = 0$  and  $L = 1$ , and solve for  $g(x)$ , and we find that:

$$\begin{aligned}
g(0) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} dx' g(x') + \sum_{n \neq 0} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^m \frac{g^{(m)}(0)}{(m+k+1)!} (\pi i n)^k (-1)^n \left(\frac{1 - (-1)^{m+k+1}}{2}\right) \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} dx' g(x') + \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^m \frac{2g^{(m)}(0) (-\pi^2)^k}{(m+2k+1)!} \left(\frac{1 - (-1)^{m+1}}{2}\right) \sum_{n=1}^{\infty} n^{2k} (-1)^n
\end{aligned} \tag{3}$$

Now the following identity is useful in terms of the Euler Polynomials:

$$\begin{aligned}
\sum_{n=1}^r (-1)^n n^{2k} &= (-1)^r \frac{E_{2k}(r+1)}{2} \\
\sum_{n=1}^{\infty} (-1)^n n^{2k} &= (2^{2k+1} - 1) \zeta(-2k) = -\frac{1}{2} \delta_{k0}
\end{aligned}$$

where we see here that the  $\zeta$  relation is the one that makes Equation 3 work. It is unclear why in some cases we want to take the Riemann Zeta analytic continuation and then other times we want to use the literal definition of the series (where of course the negative even integers of Riemann Zeta are trivial zeroes). Regardless, we then attain:

$$\begin{aligned}
g(0) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} dx' g(x') + \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^m \frac{2g^{(m)}(0) (-\pi^2)^k}{(m+2k+1)!} \left(\frac{1 - (-1)^{m+1}}{2}\right) \left(-\frac{1}{2} \delta_{k0}\right) \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} dx' g(x') + \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{2m} \frac{2g^{(2m)}(0)}{(2m+1)!} \left(-\frac{1}{2}\right) \\
&= \sum_{m=0}^{\infty} 2 \left(\frac{1}{2}\right)^{2m+1} \frac{g^{(2m)}(0)}{(2m+1)!} - \sum_{m=0}^{\infty} 2 \left(\frac{1}{2}\right)^{2m+1} \frac{g^{(2m)}(0)}{(2m+1)!}
\end{aligned} \tag{4}$$

As we can see, this calculation gives unexpected, *incorrect* results because it suggests that  $g(0) = 0$  for any  $g(x)$ . However, now we will investigate a more correct way (where it is essential that  $s$  be an integer) to verify Equation 1:

$$\gamma(s+1, z) = s! - \sum_{q=0}^s \frac{s! z^{s-q}}{(s-q)!} e^{-z}$$

Then we progress analogous to Equation 3:

$$\begin{aligned}
c_n &= \sum_{m=0}^{\infty} \sum_{q=0}^m \frac{g^{(m)}(0)}{q!} \left(\frac{1}{2}\right)^m \left(\frac{1}{i\pi n}\right)^{m-q+1} (-1)^{n-1} \left(\frac{1 - (-1)^q}{2}\right) \\
\Rightarrow g(0) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} dx' g(x') + \sum_{n \neq 0} \sum_{m=0}^{\infty} \sum_{q=0}^m \frac{g^{(m)}(0)}{q! (i\pi n)^{m-q+1}} \left(\frac{1}{2}\right)^m (-1)^{n-1} \left(\frac{1 - (-1)^q}{2}\right) \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} dx' g(x') + \sum_{n \neq 0} \sum_{m=0}^{\infty} \sum_{q=0}^{\lceil \frac{m}{2} \rceil - 1} \frac{g^{(m)}(0)}{(2q+1)!} \left(\frac{1}{2}\right)^m \left(\frac{1}{i\pi n}\right)^{m-2q} (-1)^{n-1} \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} dx' g(x') + 2 \sum_{m=0}^{\infty} \sum_{q=0}^{m-1} \frac{g^{(2m)}(0) (1 - 2^{1-2(m-q)})}{(2q+1)! (-\pi^2)^{m-q}} \left(\frac{1}{2}\right)^{2m} \zeta(2(m-q))
\end{aligned} \tag{5}$$

And my favourite form is:

$$g(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} dx' g(x') - \sum_{m=0}^{\infty} \sum_{q=0}^m \frac{g^{(2m)}(0) (1 - 2^{1-2q})}{(2(m-q)+1)!} \left(\frac{1}{2}\right)^{2(m-q)} \frac{B_{2q}}{(2q)!} - \frac{g^{(2m)}(0)}{4^m (2m+1)!} \quad (6)$$

The analysis given by Equation 5 can be generalized to:

$$\begin{aligned} g(x) &= \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx' g(x') - \sum_{n \neq 0}^{\pm\infty} \sum_{m=0}^{\infty} \sum_{q=0}^{\lceil \frac{m}{2} \rceil - 1} \frac{g^{(m)}(0)}{(2q+1)!} \left(\frac{L}{2}\right)^m \left( \frac{(-1)^n e^{\pm 2\pi i n \frac{x}{L}}}{(\pm i\pi n)^{m-2q}} \right) \\ &= \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} dx' g(x') - \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{q=0}^{\lceil \frac{m}{2} \rceil - 1} \frac{g^{(m)}(0)}{(2q+1)!} \left(\frac{L}{2}\right)^m \left( \frac{(-e^{2\pi i \frac{x}{L}})^n}{(i\pi n)^{m-2q}} + \frac{(-e^{-2\pi i \frac{x}{L}})^n}{(-i\pi n)^{m-2q}} \right) \\ &= \sum_{m=0}^{\infty} \frac{g^{(2m)}(0)}{(2m+1)!} \left(\frac{L}{2}\right)^{2m} \\ &\quad - \sum_{m=0}^{\infty} \sum_{q=1}^m \frac{g^{(2m)}(0)}{(2(m-q)+1)!} \left(\frac{L}{2}\right)^{2m} \frac{1}{(i\pi)^{2q}} \left( \text{Li}_{2q}(-e^{2\pi i \frac{x}{L}}) + \text{Li}_{2q}(-e^{-2\pi i \frac{x}{L}}) \right) \\ &\quad - \sum_{m=0}^{\infty} \sum_{q=0}^m \frac{g^{(2m+1)}(0)}{(2(m-q)+1)!} \left(\frac{L}{2}\right)^{2m+1} \frac{1}{(i\pi)^{2q+1}} \left( \text{Li}_{2q+1}(-e^{2\pi i \frac{x}{L}}) - \text{Li}_{2q+1}(-e^{-2\pi i \frac{x}{L}}) \right) \end{aligned} \quad (7)$$

where we notice that the even terms are even in  $x$  while the odd terms are odd in  $x$  (see the Polylogarithm terms for a critical view on this idea). This idea is still to be developed further, but it is interesting to see that a function *that can be modelled as* a Fourier Series is expressible as a series of Polylogarithm functions of rising order. I have checked this formula numerically. Also, it is worthwhile to note the following ideas before I complete this post:

- Jonquiere's Formula:

$$\text{Li}_n(e^{2\pi iz}) + (-1)^n \text{Li}_n(e^{-2\pi iz}) = -\frac{(2\pi i)^n}{n!} B_n(z)$$

- Euler's complex exponential function (useful for numerically evaluating the polylogarithms in Equation 7).

$$e^{iz} = \cos(z) + i \sin(z)$$

The second point was already used to numerically verify Equation 7. To address the point about Jonquiere's Identity, we get:

$$\begin{aligned} g(x) &= \sum_{m=0}^{\infty} \frac{g^{(2m)}(0)}{(2m+1)!} \left(\frac{L}{2}\right)^{2m} \\ &\quad + \sum_{m=0}^{\infty} \sum_{q=1}^m \frac{g^{(2m)}(0)}{(2(m-q)+1)!} \left(\frac{L}{2}\right)^{2m} \frac{2^{2q}}{(2q)!} B_{2q} \left(\frac{x}{L} + \frac{1}{2}\right) \\ &\quad + \sum_{m=0}^{\infty} \sum_{q=0}^m \frac{g^{(2m+1)}(0)}{(2(m-q)+1)!} \left(\frac{L}{2}\right)^{2m+1} \frac{2^{2q+1}}{(2q+1)!} B_{2q+1} \left(\frac{x}{L} + \frac{1}{2}\right) \\ &= \sum_{m=0}^{\infty} \frac{g^{(2m)}(0)}{(2m+1)!} \left(\frac{L}{2}\right)^{2m} + \sum_{(m,q)_{\text{same}}}^{\infty} \frac{g^{(m)}(0)}{(m-q+1)!} \left(\frac{L}{2}\right)^m \frac{2^q}{q!} B_q \left(\frac{x}{L} + \frac{1}{2}\right) \end{aligned} \quad (8)$$

I have verified this identity numerically for a few choices of  $x$ . The interesting aspect of this identity is that it appears to be "un-Fourier Series"-ed - numerics show that the results hold for any finite  $L$  and also model the  $g(x)$  function beyond  $x \geq \frac{L}{2}$  (ordinarily, the Fourier Series model only holds over the periodic  $-\frac{L}{2} \leq x \leq \frac{L}{2}$ ). An interesting example is to evaluate  $g(x) = e^x$ .

$$\begin{aligned}
g(x) &= \frac{2}{L} \sinh\left(\frac{L}{2}\right) \\
&+ \sum_{q=1}^{\infty} \sum_{m=q}^{\infty} \frac{\left(\frac{L}{2}\right)^{2m}}{(2(m-q)+1)! (2q)!} 2^{2q} B_{2q}\left(\frac{x}{L} + \frac{1}{2}\right) \\
&+ \sum_{q=0}^{\infty} \sum_{m=q}^{\infty} \frac{\left(\frac{L}{2}\right)^{2m+1}}{(2(m-q)+1)! (2q+1)!} 2^{2q+1} B_{2q+1}\left(\frac{x}{L} + \frac{1}{2}\right) \\
&= \frac{2}{L} \sinh\left(\frac{L}{2}\right) + 2 \sinh\left(\frac{L}{2}\right) \sum_{q=1}^{\infty} L^{2q-1} \frac{B_{2q}\left(\frac{x}{L} + \frac{1}{2}\right)}{(2q)!} + 2 \sinh\left(\frac{L}{2}\right) \sum_{q=0}^{\infty} L^{2q} \frac{B_{2q+1}\left(\frac{x}{L} + \frac{1}{2}\right)}{(2q+1)!} \quad (9) \\
&= \frac{2}{L} \sinh\left(\frac{L}{2}\right) + \frac{2}{L} \sinh\left(\frac{L}{2}\right) \sum_{q=1}^{\infty} L^q \frac{B_q\left(\frac{x}{L} + \frac{1}{2}\right)}{q!} \\
&\implies \sum_{q=1}^{\infty} \frac{L^q}{q!} B_q\left(\frac{x}{L} + \frac{1}{2}\right) = \frac{e^x - \frac{2}{L} \sinh\left(\frac{L}{2}\right)}{\frac{2}{L} \sinh\left(\frac{L}{2}\right)}
\end{aligned}$$

This relation has been checked numerically, and shows only one instance where these general series relation can produce unexpected results (in terms of the series of mysterious Bernoulli polynomials) where  $g(x)$  is fully independent of the variable  $L$ . I look forward to exploring this relation more in future posts to uncover the mystery of the Bernoulli polynomials in expressing any analytic function  $g(x)$ .

Now I search for a general truth about the relation of the Bernoulli Polynomials to the analytic function  $g(x)$ :

$$\begin{aligned}
g(x) &= \sum_{m=0}^{\infty} \frac{g^{(2m)}(0)}{(2m+1)!} \left(\frac{L}{2}\right)^{2m} \\
&+ \sum_{q=1}^{\infty} \sum_{m=0}^{\infty} \frac{g^{(2m+2q)}(0)}{(2m+1)!} \left(\frac{L}{2}\right)^{2m+2q} \frac{2^{2q}}{(2q)!} B_{2q}\left(\frac{x}{L} + \frac{1}{2}\right) \\
&+ \sum_{q=0}^{\infty} \sum_{m=0}^{\infty} \frac{g^{(2m+2q+1)}(0)}{(2m+1)!} \left(\frac{L}{2}\right)^{2m+2q+1} \frac{2^{2q+1}}{(2q+1)!} B_{2q+1}\left(\frac{x}{L} + \frac{1}{2}\right) \\
&= \sum_{m=0}^{\infty} \frac{g^{(2m)}(0)}{(2m+1)!} \left(\frac{L}{2}\right)^{2m} + \sum_{q=1}^{\infty} \sum_{m=0}^{\infty} \frac{g^{(2m+q)}(0)}{(2m+1)!} \frac{B_q\left(\frac{x}{L} + \frac{1}{2}\right)}{q!} \left(\frac{L}{2}\right)^{2m} L^q \quad (10)
\end{aligned}$$

This relation would seem to have the effect of:

$$\sum_{m=0}^{\infty} \frac{g^{(2m+q)}(0)}{(2m+1)!} L^{2m} = \frac{g^{(q)}(0) x^q}{L^q B_q\left(\frac{x}{L} + \frac{1}{2}\right)} + \frac{\delta_{q1} q!}{L^q B_q\left(\frac{x}{L} + \frac{1}{2}\right)} \left(1 - \sum_{m=0}^{\infty} \frac{g^{(2m)}(0)}{(2m+1)!} \left(\frac{L}{2}\right)^{2m}\right)$$

a relation that would give the correct numerical answer. But there is more mystery here - if we substitute  $g(x) = e^x$ , then we see that the left side of the above is independent of  $q$ , while the right

side is clearly dependent on  $q$ . Furthermore, the term-wise equality is not appropriate, though the equality holds after counting all the infinite terms. There is one more fundamental relation that I aim to examine (note that  $B'_q(x) = qB_{q-1}(x)$ ) - taking the 2 derivatives of Equation 10 (1 with respect to  $x$  and 1 with respect to  $L$ ):

$$0 = \sum_{m=0}^{\infty} \frac{L^{2m}}{2^{2m} (2m+1)!} \left( \sum_{q=1}^{\infty} \frac{(2m) B_{q-1} \left( \frac{x}{L} + \frac{1}{2} \right)}{\frac{(q-1)!}{L^{q-2} g^{(2m+q)}(0)}} + \sum_{q=2}^{\infty} \frac{(B_{q-1} \left( \frac{x}{L} + \frac{1}{2} \right) - \frac{x}{L} B_{q-2} \left( \frac{x}{L} + \frac{1}{2} \right))}{\frac{(q-2)!}{L^{q-2} g^{(2m+q)}(0)}} \right) \quad (11)$$

Some examples of this formula being implemented -  $g(x) = e^x$ :

$$\sum_{q=2}^{\infty} \frac{B_{q-1} \left( \frac{x}{L} + \frac{1}{2} \right) - \frac{x}{L} B_{q-2} \left( \frac{x}{L} + \frac{1}{2} \right)}{\frac{(q-2)!}{L^{q-2}}} = \frac{\left( \frac{2}{L} \sinh \left( \frac{L}{2} \right) - \cosh \left( \frac{L}{2} \right) \right)}{\frac{2}{L} \sinh \left( \frac{L}{2} \right)} \sum_{q=1}^{\infty} \frac{B_{q-1} \left( \frac{x}{L} + \frac{1}{2} \right)}{\frac{(q-1)!}{L^{q-2}}} \quad (12)$$

Now we can use Equation 9 to simplify the above further:

$$\sum_{q=2}^{\infty} \frac{B_{q-1} \left( \frac{x}{L} + \frac{1}{2} \right) - \frac{x}{L} B_{q-2} \left( \frac{x}{L} + \frac{1}{2} \right)}{\frac{(q-2)!}{L^{q-2}}} = \frac{\left( \frac{2}{L} \sinh \left( \frac{L}{2} \right) - \cosh \left( \frac{L}{2} \right) \right)}{2 \sinh \left( \frac{L}{2} \right)} \left( \frac{e^x}{\frac{2}{L} \sinh \left( \frac{L}{2} \right)} \right) \quad (13)$$

which I have checked numerically. Let's try  $g^{(2m+q)}(0) = 2m + q$ :

$$\begin{aligned} 0 &= \sum_{m=0}^{\infty} \frac{L^{2m}}{2^{2m} (2m+1)!} \left( \sum_{q=1}^{\infty} \frac{(2m) B_{q-1} \left( \frac{x}{L} + \frac{1}{2} \right)}{\frac{(q-1)!}{L^{q-2} (2m+q)}} + \sum_{q=2}^{\infty} \frac{(B_{q-1} \left( \frac{x}{L} + \frac{1}{2} \right) - \frac{x}{L} B_{q-2} \left( \frac{x}{L} + \frac{1}{2} \right))}{\frac{(q-2)!}{L^{q-2} (2m+q)}} \right) \\ 0 &= \left( \frac{2}{L} \sinh \left( \frac{L}{2} \right) + \frac{L}{2} \sinh \left( \frac{L}{2} \right) - \cosh \left( \frac{L}{2} \right) \right) \left( \sum_{q=1}^{\infty} \frac{B_{q-1} \left( \frac{x}{L} + \frac{1}{2} \right)}{(q-1)!} L^{q-2} \right) \\ &+ \left( \cosh \left( \frac{L}{2} \right) - \frac{2}{L} \sinh \left( \frac{L}{2} \right) \right) \left( \sum_{q=1}^{\infty} \frac{q B_{q-1} \left( \frac{x}{L} + \frac{1}{2} \right)}{(q-1)!} L^{q-2} \right) \\ &+ \left( \cosh \left( \frac{L}{2} \right) - \frac{2}{L} \sinh \left( \frac{L}{2} \right) \right) \sum_{q=2}^{\infty} \frac{(B_{q-1} \left( \frac{x}{L} + \frac{1}{2} \right) - \frac{x}{L} B_{q-2} \left( \frac{x}{L} + \frac{1}{2} \right))}{(q-2)!} L^{q-2} \\ &+ \frac{2}{L} \sinh \left( \frac{L}{2} \right) \sum_{q=2}^{\infty} \frac{q (B_{q-1} \left( \frac{x}{L} + \frac{1}{2} \right) - \frac{x}{L} B_{q-2} \left( \frac{x}{L} + \frac{1}{2} \right))}{(q-2)!} L^{q-2} \end{aligned} \quad (14)$$

where a similar technique to Equation 13 with applying Equation 10 could be used for this calculation to simplify the result (after calculating  $g(x)$  from the Taylor Series). Before ending this post, it would be interesting to examine the implications of Equation 11, where we have multiple variables and an entire function space that always yield the same result - 0. It would be interesting to consider the implication of using this result in an algebraic context as a  $\pm 0$  vanishing function that then could interact and change the form of the current algebra. I will continue to think about such an application. This concludes this blog post.