

Title: Stieltjes Constants and Riemann Zeta
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May 16, 2026

The Laurent Series for the Riemann Zeta function is:

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n (s-1)^n \quad (1)$$

for $\text{Re}(s) \geq 0$, where γ_n is the n th Stieltjes constant, defined as:

$$\gamma_n = \lim_{m \rightarrow \infty} \left[\sum_{k=1}^m \frac{\ln^n(k)}{k} - \frac{\ln^{n+1}(m)}{n+1} \right] = \lim_{z \rightarrow 1} \left[(-1)^n \zeta^{(n)}(z) - \frac{n!}{(z-1)^{n+1}} \right] \quad (2)$$

where note that the second equality follows directly from Equation 1. Note the following identities:

$$\begin{aligned} \ln^n(k) &= \lim_{x \rightarrow 0} \left(\frac{k^x - 1}{x} \right)^n \\ \lfloor n \rfloor &= \lim_{\alpha \rightarrow 0} \partial_{\alpha}^n \frac{(\alpha)_r}{n!} \\ H_n^{(\omega)} &= \sum_{k=1}^n \frac{1}{k^{\omega}} = \frac{1}{\Gamma(\omega)} \int_0^1 dx \frac{1-x^n}{1-x} \ln^{\omega-1} \left(\frac{1}{x} \right) \end{aligned} \quad (3)$$

Now we consider the first equality:

$$\begin{aligned} \gamma_n &= \lim_{m \rightarrow \infty} \lim_{x \rightarrow 0} \left[\sum_{k=1}^m \frac{(k^x - 1)^n}{x^n k} - \frac{\ln^{n+1}(m)}{(n+1)} \right] \\ &= \lim_{m \rightarrow \infty} \lim_{x \rightarrow 0} \left[\sum_{k=1}^m \sum_{q=0}^n \binom{n}{q} \frac{k^{qx-1} (-1)^{n-q}}{x^n} - \frac{(-\ln(\frac{1}{m}))^{n+1}}{(n+1)} \right] \\ &= \lim_{m \rightarrow \infty} \lim_{x \rightarrow 0} \left[\sum_{q=0}^n \binom{n}{q} \frac{(-1)^{n-q}}{x^n} H_m^{(1-qx)} - \sum_{r=n+1}^{\infty} \lfloor n+1 \rfloor \frac{n!}{r!} \left(1 - \frac{1}{m} \right)^r \right] \\ &= \lim_{m \rightarrow \infty} \lim_{x \rightarrow 0} \left[\sum_{q=0}^n \binom{n}{q} \frac{(-1)^{n-q}}{x^n} H_m^{(1-qx)} \right] - \left[\sum_{r=n+1}^{\infty} \frac{\lim_{\alpha \rightarrow 0} \partial_{\alpha}^{n+1} (\alpha)_r}{r! (n+1)} \left(1 - \frac{1}{m} \right)^r \right] \end{aligned} \quad (4)$$

Note that the order of summation and differentiation in the second term in the bottom line cannot be changed because the sum diverges. Also, the harmonic number term cannot be converted to a

logarithmic harmonic series, since the small differences resulting from the qx term makes the full difference between diverging and converging results. The result can be given eloquently as:

$$\gamma_n = \lim_{m \rightarrow \infty} \lim_{x \rightarrow 0} \left[\frac{1}{x^n} \sum_{q=0}^n \binom{n}{q} (-1)^{n-q} H_m^{(1-qx)} - \frac{(\ln(m))^{n+1}}{n+1} \right] \quad (5)$$

Before proceeding with the commentary, it is noteworthy that Mathematica suggests that:

$$\lim_{m \rightarrow \infty} \lim_{x \rightarrow 0} \frac{1}{x^n} \sum_{q=0}^n \binom{n}{q} (-1)^{n-q} H_m^{(1-qx)} = \gamma_n - \lim_{m \rightarrow \infty} \gamma_n(m+1) \quad (6)$$

where $\lim_{m \rightarrow \infty} \gamma_n(m+1) = -\lim_{m \rightarrow \infty} \frac{\log^{n+1}(m)}{n+1}$ is the generalized Stieltjes constant, corresponding to:

$$\zeta(s, a) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{\gamma_n(a)}{n!} (1-s)^n$$

where $\zeta(s, a)$ is the Hurwitz Zeta function. Before progressing forward, we will quickly show that Equation 2 and Equation 5 are intuitively connected by L'Hopital's Rule:

$$\lim_{x \rightarrow 0} \frac{\lim_{m \rightarrow \infty} H_m^{(1-qx)}}{x^n} = \frac{1}{n!} \sum_{k=1}^m \frac{q^n \ln^n(k)}{k} \quad (7)$$

and the two formulas are then connected precisely by the following realization:

$$\sum_{q=0}^n \frac{q^n (-1)^{n-q}}{q! (n-q)!} = 1, \quad n \forall \in \mathbb{W} \quad (8)$$

Now that we have confirmed that Equation 5 and Equation 2 are equivalent, we can proceed using the integral identity for the Harmonic Number function in Equation 3:

$$\gamma_n = \lim_{m \rightarrow \infty} \left[\sum_{q=0}^n \binom{n}{q} (-1)^{n-q} \lim_{x \rightarrow 0} \left(\int_0^1 dt \frac{1-t^m}{1-t} \frac{(-\ln(t))^{-qx}}{\Gamma(1-qx)} \right) - \frac{(\ln(m))^{n+1}}{n+1} \right] \quad (9)$$

Now let's consider $n = 1$, where we use the following identity:

$$\lim_{x \rightarrow 0} \frac{\frac{(-\ln(t))^{-qx}}{\Gamma(1-qx)}}{x} = -q(\gamma_0 + \ln(-\ln(t)))$$

Then:

$$\gamma_1 = \lim_{m \rightarrow \infty} \left[- \int_0^1 dt \frac{1-t^m}{1-t} (\gamma_0 + \ln(-\ln(t))) - \frac{(\ln(m))^2}{2} \right] \quad (10)$$

For $n = 2$:

$$\lim_{x \rightarrow 0} \frac{\frac{(-\ln(t))^{-qx}}{\Gamma(1-qx)}}{x^2} = \frac{q^2}{2} \left(-\frac{\pi^2}{6} + (\gamma_0 + \ln(-\ln(t)))^2 \right)$$

Thus, γ_2 :

$$\gamma_2 = \lim_{m \rightarrow \infty} \left[\int_0^1 dt \frac{1-t^m}{1-t} \left(-\frac{\pi^2}{6} + (\gamma_0 + \ln(-\ln(t)))^2 \right) - \frac{(\ln(m))^3}{3} \right] \quad (11)$$

where the sum vanishes because of Equation 8. This concludes this blog post.