Title: Observations of Logarithmic Integrals
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June 17, 2025

In a previous post on this blog, I have investigated the solution of the following integral:

$$I_{a,b} = \int_0^1 \ln^a(x)\ln^b(1-x)\,\mathrm{d}x \tag{1}$$

Observe that the following limit corresponds to a function of the logarithmic power:

$$\lim_{r\to 0}\left(\frac{x^r-1}{r}\right) = \ln(x) \tag{2}$$

This statement is true due to application of L'Hopital's Rule. Now imagine that one could exchange the order of the limit and integration. Then one would obtain a simple function to integrate and then limit for an expression of $I_{a,b}$. It turns out that one can very often (and potentially always, if one examines the dominated convergence theorem) exchange the order of integration and limiting in this context for the logarithmic function. Now consider the following integral:

$$I_1(x) = \int_0^x \mathrm{d}t\,\ln\left(\frac{x-t}{1-t}\right)\ln^2(1-t) \tag{3}$$

for $x \leq 1$. This is a fairly intimidating integral, and requesting a solution from Mathematica or some other solver yields an expression that is very long an unconcise. However, using the exchange of limits and integration, we can re-express $I(x)$ as the following:

$$I_1(x) = \lim_{p\to 0}\lim_{r_1\to 0}\lim_{r_2\to 0}\frac{1}{p\,r_1\,r_2}\int_0^x\left(\left(\frac{x-t}{1-t}\right)^p - 1\right)\left((1-t)^{r_1}-1\right)\left((1-t)^{r_2}-1\right) \tag{4}$$

As can be seen, the resultant integral factors out into 8 terms. Carrying out integration on each term, then evaluating each limit on the full integrated result multiplied by $\frac{1}{p\,r_1\,r_2}$ yields a relatively short result composed of sophisticated hypergeometric functions:

$$I_1(x) = x\left((-\gamma+\ln(x))\,\tilde{F}_{2,1}^{(0,2,0,0)}(1,0;2;x) + \tilde{F}_{2,1}^{(0,2,1,0)}(1,0;2;x) + \tilde{F}_{2,1}^{(0,3,0,0)}(1,0;2;x)\right) \tag{5}$$

where γ is the Euler-Mascheroni constant and $\tilde{F}_{2,1}^{(m,n,s,v)}(a,b;c;x)$ is the derivatives of the regularized Gauss hypergeometric function, defined as:

$$\tilde{F}_{2,1}^{(m,q,s,v)}(a,b;c;x) = \frac{\partial^m}{\partial a^m}\frac{\partial^q}{\partial b^q}\frac{\partial^s}{\partial c^s}\frac{\partial^v}{\partial x^v}\left(\frac{1}{\Gamma(c)}\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n}\frac{x^n}{n!}\right) \tag{6}$$

where the derivatives must be taken before evaluating the values of a, b, c, and x. This example where Mathematica has been guided towards a compact answer is particularly interesting as the size of the problem grows. Some general notes about implementing this algorithm are:

- When evaluating the integral in Mathematica, it can be helpful to expand the problem and integrate each term separately. This seems to be necessary to speed up the computation process - perhaps Mathematica gets confused about how to deal with all the terms at once?
- When evaluating the limits after integration, make sure to add the entire expansion (each integrated term) together before taking the limits - if this is not done, then the limits will produce undefined answers (the integrated terms cancel to produce the correct answer when the limit is done on *all* the terms).
- When carrying out the limits, there are times when Mathematica finds evaluating the written limit (the integrated terms divided by the variable) as faster, while there are times (such as in Equation 5) where the limit is computationally faster if you take the derivative of the sum of the integrated terms with respect to the limit variable and then evaluate that limit (implementing L'Hopital's Rule).

This algorithm is quite efficient, if a bit cumbersome to enter into Mathematica when integrating *each* term. However, it leads to quite compact results that are easier to present to readers, as the hypergeometric function provides a sophisticated way of representing these challenging integrals. This algorithm is also quite general, as many integrals of integer powers of the natural logarithm with many styles of arguments can be evaluated. I encourage you to try the algorithm described in this demonstration blog post yourself - remember to make one variable for each logarithm! This concludes this blog post.