Title: Fractional Calculus: Free Parameter Equation
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Consider the following fractional calculus formulas, the second one courtesy of Podlubny:

$$\begin{aligned} \left(J^{\alpha} f\right)(x) &= \frac{1}{\Gamma(\alpha)} \int_0^1 \mathrm{d}t\ (1-t)^{\alpha-1} x^{\alpha} f(xt),\ \alpha > 0 \\ D_x^{\alpha}\left(f(x)\, g(x)\right) &= \sum_{j=0}^{\infty} \binom{\alpha}{j} \left[D_x\left(J^{1-\alpha+j} f(t)\right)(x)\right] \left[D_x^j g(x)\right],\ 0 \le \alpha \le 1 \end{aligned} \tag{1}$$

Now if we apply the second line of Equation 1 to the first line of Equation 1, then we get:

$$f(x) = \alpha \int_0^1 \mathrm{d}t\ (1-t)^{\alpha-1} \sum_{j=0}^{\infty} \frac{\binom{\alpha}{j}}{j!} f^{(j)}(xt)\,(xt)^j,\ 0 \le \alpha \le 1 \tag{2}$$

This formula is particularly interesting, as the right side must be independent of α for a general function f (since the left side is independent of α), though there is no obvious way to simplify and cancel the α dependence. Now we can show some very interesting results. Start with $f(u) = e^{-u^2}$, $f^{(j)}(u) = 2^j \sqrt{\pi} u^{-k} \tilde{F}_{22}\left(\frac{1}{2}, 1; 1 - \frac{j+1}{2}, 1 - \frac{j}{2}; -u^2\right)$, where $\tilde{F}_{22}$ is the regularized hypergeometric function. This means we can isolate for the sum:

$$\sum_{j=0}^{\infty} \left(\frac{2^{j-\alpha}}{j!} \alpha\pi \binom{\alpha}{j} \Gamma(\alpha)\, \tilde{F}_{44}\left(\frac{1}{2}, \frac{1}{2}, 1, 1; \frac{1+\alpha}{2}, \frac{2+\alpha}{2}, \frac{1-j}{2}, \frac{2-j}{2}; -x^2\right)\right) = e^{-x^2} \tag{3}$$

This means that independent of the value of α, the sum is e^{-x^2}. Now if we expand in powers of x^2, we get:

$$\begin{aligned} &\sum_{j=0}^{\infty} \frac{2^{j-\alpha}}{j!} \alpha\pi \binom{\alpha}{j} \Gamma(\alpha)\, \tilde{F}_{44}\left(\frac{1}{2}, \frac{1}{2}, 1, 1; \frac{1+\alpha}{2}, \frac{2+\alpha}{2}, \frac{1-j}{2}, \frac{2-j}{2}; -x^2\right) \\ &\quad = \sum_{n=0}^{\infty} \left(\frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n n!\, n!}{\left(\frac{1+\alpha}{2}\right)_n \left(\frac{2+\alpha}{2}\right)_n} \sum_{j=0}^{\infty} \frac{\frac{2^{j-\alpha}}{j!} \alpha\pi \binom{\alpha}{j} \Gamma(\alpha)}{\left(\frac{1-j}{2}\right)_n \left(\frac{2-j}{2}\right)_n} \frac{\frac{1}{\Gamma\left(\frac{1+\alpha}{2}\right)} \frac{1}{\Gamma\left(\frac{2+\alpha}{2}\right)}}{\Gamma\left(\frac{1-j}{2}\right) \Gamma\left(\frac{2-j}{2}\right)} \frac{\left(-x^2\right)^n}{n!} \right) \\ &\quad = \sum_{n=0}^{\infty} \frac{\left(-x^2\right)^n}{n!} \end{aligned} \tag{4}$$

Equating powers of $-x^2$, we get:

$$\left(\frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n n!\, n!}{\left(\frac{1+\alpha}{2}\right)_n \left(\frac{2+\alpha}{2}\right)_n} \sum_{j=0}^{\infty} \frac{\frac{2^{j-\alpha}}{j!} \alpha\pi \binom{\alpha}{j} \Gamma(\alpha)}{\left(\frac{1-j}{2}\right)_n \left(\frac{2-j}{2}\right)_n} \frac{\frac{1}{\Gamma\left(\frac{1+\alpha}{2}\right)} \frac{1}{\Gamma\left(\frac{2+\alpha}{2}\right)}}{\Gamma\left(\frac{1-j}{2}\right) \Gamma\left(\frac{2-j}{2}\right)} \right) = 1 \tag{5}$$

which is true for all integers $n > 0$ and real numbers $0 \leq \alpha \leq 1$. Now it is also true that if we integrate Equation 5 with respect to α from 0 to 1, then we ought to get no change in the right side and a change depending on n and j on the left side. If we do this integration before integrating in t and summing in j, we get the following:

$$\int_0^1 \mathrm{d}t \left(\frac{\ln(1-t) + \frac{t}{1-t}}{\ln^2(1-t)} f(xt) + \sum_{j=1}^{\infty} \sum_{k=0}^{j-1} \frac{\left[{j \atop k+1} \right]}{(j!)^2} \frac{f^{(j)}(xt)\, \gamma(k+3, -\ln(1-t))}{(-1)^j \ln^{k+3}(1-t)(1-t)} (xt)^j \right) = f(x) \quad (6)$$

where γ is the upper incomplete gamma function, and $\left[{j \atop k+1} \right]$ is the unsigned Stirling Numbers of the first kind. Evidently, there is utility and simplification that comes with incorporating α into the formula. In the next post, we will explore a plethora of relationships that can be derived from these calculations. This concludes this blog post.