

Title: Confined Random Triangles

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In this post, I will examine the statistics of random triangles confined to a circle of radius R . The algorithm that I used for generating these triangles is given in Figure 1. This algorithm makes use

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22 - for expcount = 1 : experimentN
23 -     avAVal = 0; AValvec = zeros(N,1); rValvec = zeros(N,1);
24 -     for count = 1 : N
25 -         xVal = -R + 2*R*rand;
26 -         yVal = (-sqrt(R^2-xVal^2) + 2*sqrt(R^2-xVal^2)*rand);
27 -         rVal = sqrt(xVal^2 + yVal^2);
28 -         thetal = -AngleMax + 2*AngleMax*rand;
29 -         yintercept2 = yVal - tan(thetal)*xVal;
30 -         xinter1 = -sin(thetal).*cos(thetal)*yintercept2 + sqrt(sin(thetal)^2*cos(thetal)^2*yintercept2^2 + (R^2-yintercept2^2)*cos(thetal)^2);
31 -         xinter2 = -sin(thetal).*cos(thetal)*yintercept2 - sqrt(sin(thetal)^2*cos(thetal)^2*yintercept2^2 + (R^2-yintercept2^2)*cos(thetal)^2);
32 -         yinter1 = tan(thetal)*xinter1 + yintercept2;
33 -         yinter2 = tan(thetal)*xinter2 + yintercept2;
34 -         aVal1 = sqrt((xinter1-xVal)^2+(yinter1-yVal)^2)*rand;
35 -         aVal2 = sqrt((xinter2-xVal)^2+(yinter2-yVal)^2)*rand;
36 -         aVal = aVal1 + aVal2;
37 -         xVal1 = aVal1*cos(thetal) + xVal; yVal1 = aVal1*sin(thetal) + yVal;
38 -         xVal2 = -aVal2*cos(thetal) + xVal; yVal2 = -aVal2*sin(thetal) + yVal;
39 -         alpha = -AngleMax + 2*AngleMax*rand;
40 -         dirNm = (-1)^randi(2);
41 -         yintercept = yVal1 - xVal1*tan(alpha);
42 -         xinterVal2 = -sin(alpha).*cos(alpha)*yintercept + dirNm*sqrt(sin(alpha)^2*cos(alpha)^2*yintercept^2 + (R^2-yintercept^2)*cos(alpha)^2);
43 -         yinterVal2 = tan(alpha)*xinterVal2 + yintercept;
44 -         bVal = sqrt((xinterVal2-xVal1)^2+(yinterVal2-yVal1)^2)*rand;
45 -         AVal = abs(1/2*aVal*bVal*sin(pi-thetal+alpha));
46 -         xVal21 = dirNm*bVal*cos(alpha) + xVal1; yVal21 = dirNm*bVal*sin(alpha) + yVal1;
47 -         cVal = sqrt((xVal21-xVal2)^2+(yVal21-yVal2)^2);
48 -         avAVal = (count - 1)/count*avAVal + AVal/count;

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Figure 1: The algorithm for generating truly random triangles confined to a circle with radius R .

of a variety of geometric ideas and relative coordinate systems. The following list highlights some of the main ideas used to obtain the algorithm:

- The slope of a line can be expressed as $\tan(\theta)$ for a θ defined as the angle between the horizontal x-axis and the line in question. This idea has origin related to the unit circle, where a point on the circle has coordinates $(x, y) = (\cos(\theta), \sin(\theta))$.
- The algorithm ensures confinement within the circle by calculating the two points of intersection with the circle of a straight line passing through a point confined within the circle. Then, the side length of the triangle is generated by multiplying a uniformly distributed random number by the maximum length between the point in question and the POI of the circle. In particular, the following POI formula is useful for a line defined by $y = \tan(\theta)x + y_0$, where $y_0 = y_i - \tan(\theta)x_i$, where (x_i, y_i) is one of the vertices of the triangle (or at least a point of interest, such as $(xVal, yVal)$ variable in the above computation:

$$x_{POI_{1,2}} = -\sin(\theta)\cos(\theta)y_0 \pm \sqrt{\sin^2(\theta)\cos^2(\theta)y_0^2 + (R^2 - y_0^2)\cos^2(\theta)} \quad (1)$$

This statement of the solution is intuitive since it keeps the discriminant in a similar form to the term outside the square root. Note that y_0 need not necessarily be confined inside

the circle, as long as the side length of the triangle to which the line corresponds is confined within the circle.

- The choice to define an “xVal” and “yVal” in the first two lines that do not correspond to a vertex of the triangle (but instead act as a point of intersection for the first side length a) was done such that a conditional probability analysis could be done to compare the distributions of triangle area with initial radius from the centre of the circle.

After creating the algorithm to generate the random triangles, I went to work with analysing the probability distributed data. First off, it seems a central motivation to study the influence of the circular confinement. Thus, I examined the average area of the triangle vs. the starting radius of the triangle (“rVal” in Figure 1). From this data, it becomes clear that as we move towards starting near the outer radius, the average area of the triangle generated in this area is less than the triangle at radius near 0. So spatial confinement of the triangle to within the circle has a real effect on the average area of the triangle. The distribution results are in Figure 2 for $N = 10^9$ iterations and $R = 1$. The global average area for all radii is $A_{av} \approx 0.103$, a result that seems to scale with radius as $A_{av} \sim 0.103R^2$, which makes sense as the dimensions are randomized relative to the points of intersection with the circle, so growing the circle radius will on average lead to larger side lengths to the extent of producing an average area that grows with the square of R . However, this is not the full story, as if we look at the conditional probability distribution for triangles generated near the outer radius, we see a bimodal cumulative probability distribution where there are relatively more triangles with very small area (confinement) but also relatively more triangles with large area. This property is a result of the “dirNm” variable when generating the second side length we call b in Figure 1. These conditional probability ratio results (see Equation 2) for $r = 0R$ and $r = 0.98R$ are given Figure 3 and Figure 4, respectively, for $N = 10^9$ iterations.

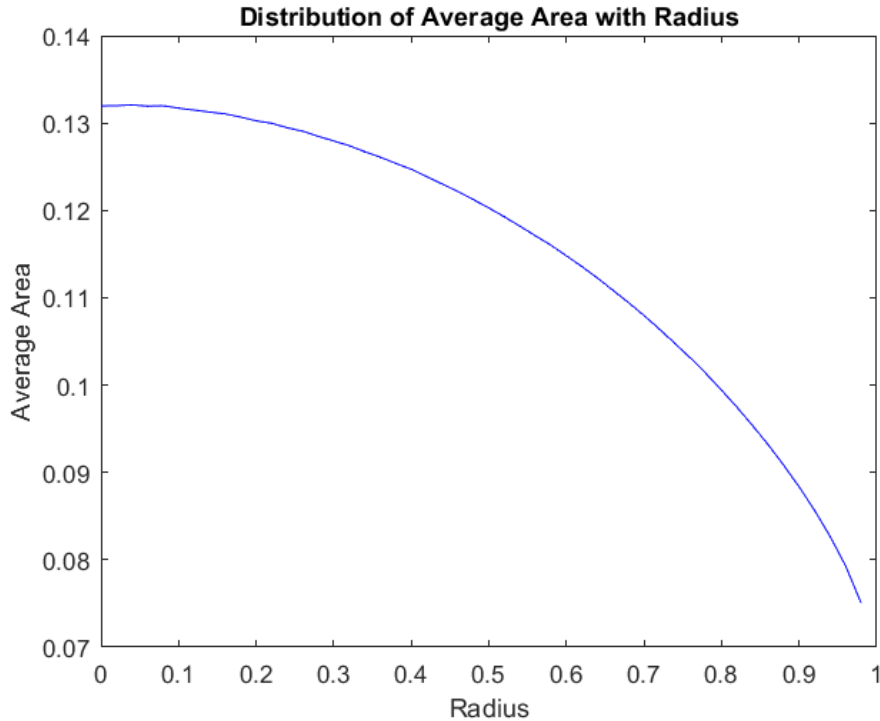


Figure 2: The distribution of average area vs. starting radius (“rVal”) for $N = 10^9$ and $R = 1$.

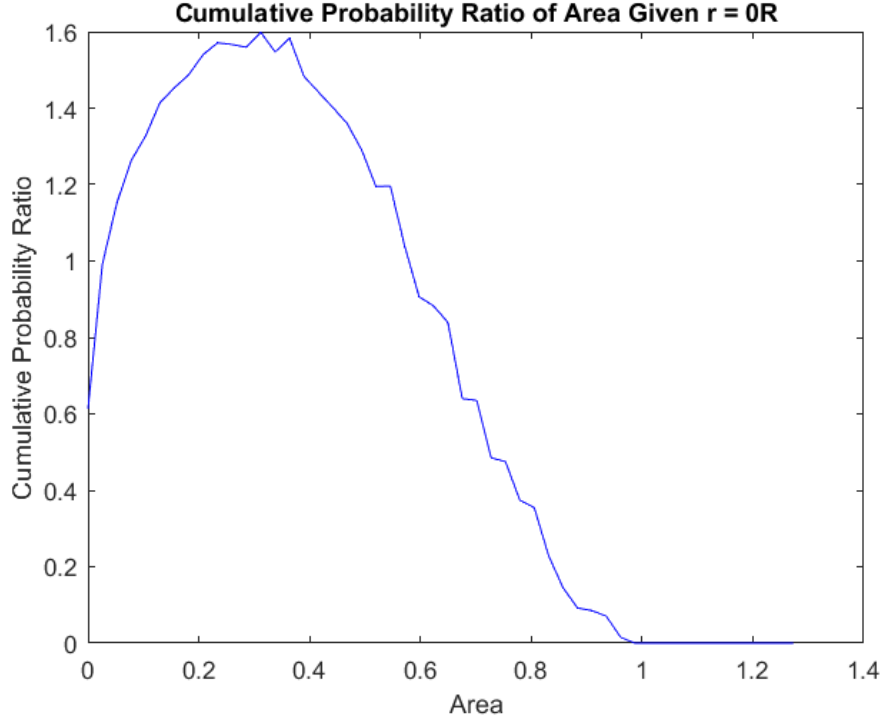


Figure 3: The probability ratio for $0R \leq r \leq 0.02R$ for $N = 10^9$ iterations and $R = 1$.

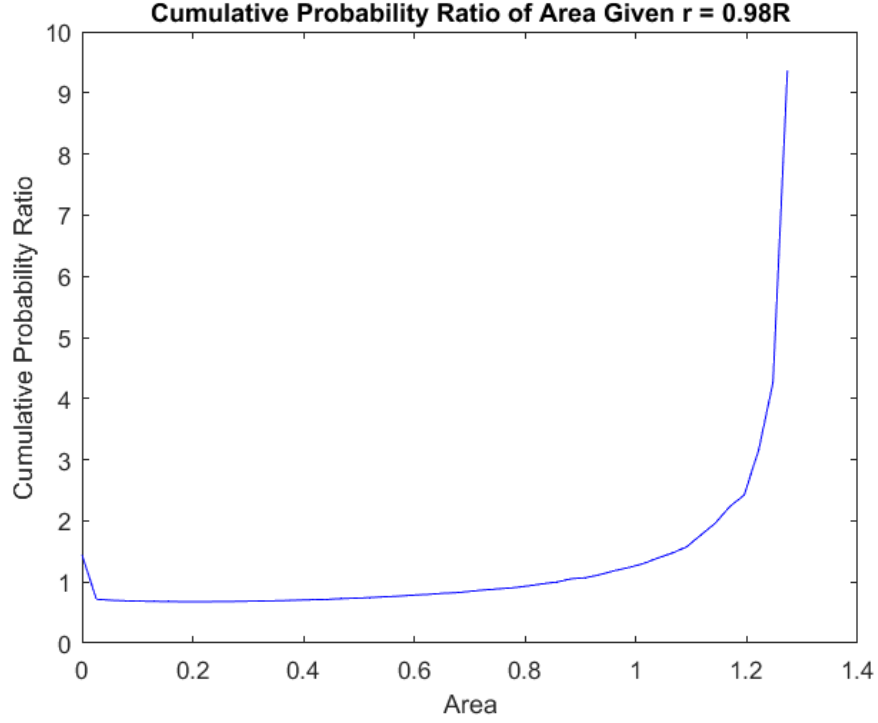


Figure 4: The probability ratio for $0.98R \leq r \leq 1R$ for $N = 10^9$ iterations and $R = 1$.

The probability ratio in Figure 3 and Figure 4 is defined by the following equation:

$$P_{\text{ratio}} = \frac{P(A|r)}{P(A)} \quad (2)$$

When the probability ratio is greater than 1, this means that the number of triangles in a A bin given an r is greater than the global average of A for any r . On the contrary, if the probability ratio is less than 1, then the number of triangles with an A given an r is less than the global average of A for any r . These interpretations bring meaning to the probability ratio - it represents the extent that the A conditional distribution is over- or under-represented relative to the global A distribution. Several noteworthy comments come out of this data - first, that constraining the initial radius point to be near the centre of the circle ($0R \leq r \leq 0.02R$) naturally limits the total area to which the triangle can grow. Second, given that the area represented by triangles at the centre of the circle is less, the data is far more noisy as fewer Monte Carlo cases produced this “rVal” relative to higher radii. The data is far less noisy for the $0.98R \leq r \leq 1R$ case, but does get much noisier near higher area bins due to the limited number of iterations where the random triangle grew that large. Note also that the proportion of small area triangles is also greater than 1 for Figure 4, meaning that the proportion of smaller triangles at this radius is greater relative to the global average probability also (a case where confinement within the circle seems to play a role). But the probability ratio at higher areas in Figure 4 shows clearly that there is a larger proportion of high area cases for triangles starting positioned at higher radius.

We will conclude with a 3 point examination of conditional probability in the form of Bayes’ Theorem:

- The theoretical cumulative probability distribution $P(r_1 \leq r \leq r_2)$ can be expressed as the uniformly distributed circle area:

$$P(r_1 \leq r \leq r_2) = \frac{r_2^2 - r_1^2}{R^2} \quad (3)$$

- Bayes’ Theorem can be expressed as:

$$P(A|r) = \frac{P(r|A) P(A)}{P(r)} \quad (4)$$

This expression of Bayes’ Theorem gives the following interesting matrix expression:

$$\frac{P(A_i|r_j)}{P(r_j|A_i)} = \begin{bmatrix} P(A_1) \\ P(A_2) \\ \vdots \\ P(A_N) \end{bmatrix} \begin{bmatrix} \frac{1}{P(r_1)} & \frac{1}{P(r_2)} & \cdot & \cdot & \frac{1}{P(r_M)} \end{bmatrix} \quad (5)$$

which leads to a $N \times M$ matrix of probabilities. In this way, one can learn about the ratio of conditional probabilities by knowing only the global distribution of area and radius individually. This then leads us to our third point, for which we may investigate further at a later time.

- The area distribution of triangles may be found through the following formula:

$$P(A_i) = \frac{P(A_i|r_j)}{P(r_j|A_i)} P(r_j) \quad (6)$$

In future, I will look at if the conditional ratio can be useful. This concludes this blog post.