Title: Computing the Non-Trivial Zeroes of Riemann Zeta on Critical Strip
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One form of the Riemann-Zeta functional equation is:

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta\left(s\right)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta\left(1-s\right) \tag{1}$$

This form naturally leads to the following functional equation:

$$\frac{\zeta\left(s\right)}{\zeta\left(1-s\right)}=\pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \tag{2}$$

On the critical strip, where $s=\frac{1}{2}+it$, we get:

$$\frac{\zeta\left(\frac{1}{2}+it\right)}{\zeta\left(\frac{1}{2}-it\right)}=\pi^{it}\frac{\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)} \tag{3}$$

The left and right sides of Equation 3 have absolute values of 1. Thus, it is reasonable to write the equality in terms of a complex phase - let:

$$\begin{gathered}e^{-2i\theta(t)}=\pi^{it}\frac{\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)}\\ \implies e^{i\theta(t)}\zeta\left(\frac{1}{2}+it\right)=e^{-i\theta(t)}\zeta\left(\frac{1}{2}-it\right)\end{gathered} \tag{4}$$

where the left and right sides are complex conjugates of one another. The concept of this equality implies that both the left and the right sides of Equation 4 are real, as this is the only possible implication of a number and its complex conjugate being equal. With this said, we can now define the Hardy Z function, which has zeroes for the same t input (only along the critical strip $s=\frac{1}{2}+it$) as the Riemann Zeta but is additionally real for all values of t. This realization makes the computation of non-trivial zeroes (along the critical strip) significantly easier, as we need only track one variable t and one (real) function output looking for roots, rather than two function outputs (real and complex) for the parameter input t for where they are *both* zero.

$$Z\left(t\right)=e^{i\theta(t)}\zeta\left(\frac{1}{2}+it\right)=\pi^{-\frac{it}{2}}\frac{\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)}{\left|\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)\right|}\zeta\left(\frac{1}{2}+it\right) \tag{5}$$

Now, we can compute the Hardy Z function, assured that the situation is a one variable input and one real output, allowing us to plot the result in Figure 1.

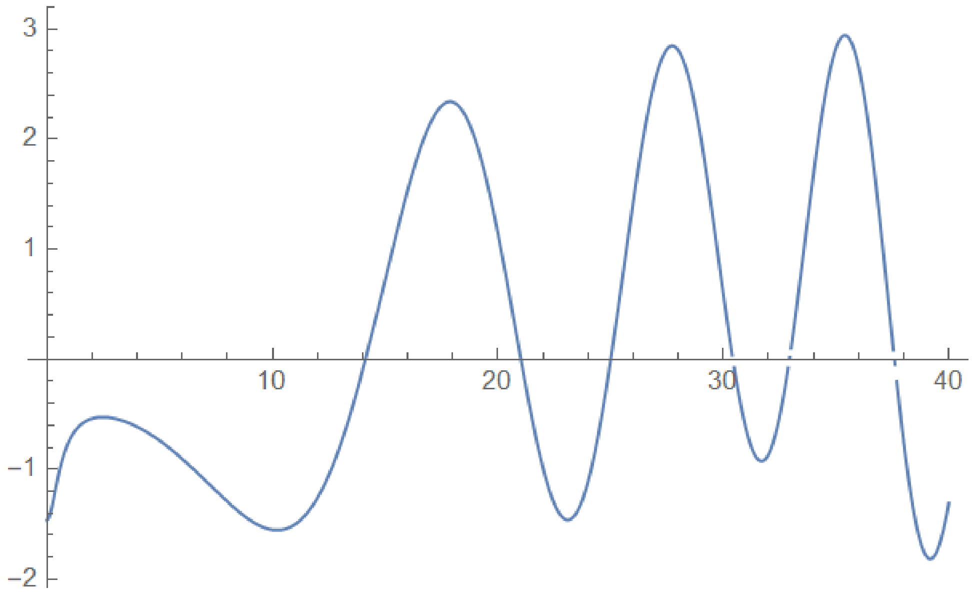

Figure 1: The first 6 non-trivial zeroes of the Riemann Zeta function along the critical line. Note that $|Z(t)| = \left|\zeta\left(\frac{1}{2}+it\right)\right|$.

Now consider points off the critical line - $s = \frac{1}{2}+\delta+it$. We would attempt to study this case using the Taylor expansion. To do this, we will re-write the function in terms of the general specified s. Re-visit Equation 2:

$$\frac{\zeta\left(\frac{1}{2}+\delta+it\right)}{\zeta\left(\frac{1}{2}-\delta-it\right)} = \pi^{\delta+it}\frac{\Gamma\left(\frac{1}{4}-\frac{\delta}{2}-\frac{it}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{\delta}{2}+\frac{it}{2}\right)} \tag{6}$$

Now consider $\delta \to 0$ - we then consider the linear term of the Taylor expansion. The desired derivative is:

$$\begin{aligned}\left(\partial_\delta \pi^{\delta+it}\frac{\Gamma\left(\frac{1}{4}-\frac{\delta}{2}-\frac{it}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{\delta}{2}+\frac{it}{2}\right)}\right)_{\delta\to 0} &= \left(\ln(\pi)\,\pi^{\delta+it}\frac{\Gamma\left(\frac{1}{4}-\frac{\delta}{2}-\frac{it}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{\delta}{2}+\frac{it}{2}\right)} - \frac{\pi^{\delta+it}}{2}\frac{\Gamma'\left(\frac{1}{4}-\frac{\delta}{2}-\frac{it}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{\delta}{2}+\frac{it}{2}\right)}\right)_{\delta\to 0} \\ &\quad - \left(\frac{\pi^{\delta+it}}{2}\frac{\Gamma\left(\frac{1}{4}-\frac{\delta}{2}-\frac{it}{2}\right)\Gamma'\left(\frac{1}{4}+\frac{\delta}{2}+\frac{it}{2}\right)}{\left(\Gamma\left(\frac{1}{4}+\frac{\delta}{2}+\frac{it}{2}\right)\right)^2}\right)_{\delta\to 0} \\ &= \ln(\pi)\,\pi^{it}\frac{\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)} \\ &\quad - \frac{\pi^{it}}{2}\frac{\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)}\left(\psi_0\left(\frac{1}{4}-\frac{it}{2}\right)+\psi_0\left(\frac{1}{4}+\frac{it}{2}\right)\right)\end{aligned} \tag{7}$$

where $\psi\left(\frac{1}{4}+\frac{it}{2}\right)$ is the 0th order polygamma function. Thus:

$$\begin{aligned}\lim_{\delta\to 0}\frac{\zeta\left(\frac{1}{2}+\delta+it\right)}{\zeta\left(\frac{1}{2}-\delta-it\right)} &= \pi^{it}\frac{\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)} + \ln(\pi)\,\pi^{it}\frac{\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)}\delta \\ &\quad - \frac{\pi^{it}}{2}\frac{\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)}\left(\psi_0\left(\frac{1}{4}-\frac{it}{2}\right)+\psi_0\left(\frac{1}{4}+\frac{it}{2}\right)\right)\delta\end{aligned} \tag{8}$$

Though, evidently this equation does not preserve the magnitude as Equation 2 does, as for any finite δ, there will be a departure from the ratio of the ζ functions being 1, and furthermore we will not have the nice algebraic symmetry that gives rise to Hardy's Z-function that is purely real-valued. However, returning to Equation 7, it seems potentially plausible to write down a general term for the nth derivative $\left(D_\delta^n f_{\delta,t}\right)_{\delta\to 0}$ with respect to δ, where the expression for $n=1$ is represented by the left side of Equation 7. We write down $n=2$ below as:

$$\begin{aligned}
\left(D_\delta^0 f_{\delta,t}\right)_{\delta\to 0} &= \pi^{it}\frac{\Gamma\left(\frac{1}{4}-\frac{it}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)} \\
\left(D_\delta^1 f_{\delta,t}\right)_{\delta\to 0} &= \left(D_\delta^0 f_{\delta,t}\right)_{\delta\to 0}\left(\ln(\pi)-\frac{1}{2}\left(\psi_0\left(\frac{1}{4}+\frac{it}{2}\right)+\psi_0\left(\frac{1}{4}-\frac{it}{2}\right)\right)\right) \\
\left(D_\delta^2 f_{\delta,t}\right)_{\delta\to 0} &= \left(D_\delta^1 f_{\delta,t}\right)_{\delta\to 0}\left(\ln(\pi)-\frac{1}{2}\left(\psi_0\left(\frac{1}{4}+\frac{it}{2}\right)+\psi_0\left(\frac{1}{4}-\frac{it}{2}\right)\right)\right) \\
&\quad -\frac{1}{4}\left(D_\delta^0 f_{\delta,t}\right)_{\delta\to 0}\left(\psi_1\left(\frac{1}{4}+\frac{it}{2}\right)-\psi_1\left(\frac{1}{4}-\frac{it}{2}\right)\right) \\
\left(D_\delta^n f_{\delta,t}\right)_{\delta\to 0} &= \sum_{k=0}^{n-1}\binom{n-1}{k}\left(D_\delta^{n-1-k} f_{\delta,t}\right)_{\delta\to 0} \\
&\qquad \left(\ln(\pi)\,\delta_{k0}-\frac{1}{2^{k+1}}\left(\psi_k\left(\frac{1}{4}+\frac{it}{2}\right)+(-1)^k\,\psi_k\left(\frac{1}{4}-\frac{it}{2}\right)\right)\right)
\end{aligned} \tag{9}$$

where δ_{k0} is the Kronecker Delta function. Now we can write Equation 6 as a Taylor series:

$$\frac{\zeta\left(\frac{1}{2}+\delta+it\right)}{\zeta\left(\frac{1}{2}-\delta-it\right)} = \sum_{n=0}^{\infty}\frac{\left(D_\delta^n f_{\delta,t}\right)_{\delta\to 0}}{n!}\delta^n,\ f_{\delta,t} = \pi^{\delta+it}\frac{\Gamma\left(\frac{1}{4}-\frac{\delta}{2}-\frac{it}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{\delta}{2}+\frac{it}{2}\right)} \tag{10}$$

This identity has been confirmed numerically for various values of $|t|=1$ and $|\delta|<\frac{1}{2}$. Note that for $\delta=-1/2$ and $t=0$, the identity fails badly, as the Taylor Series does not exist for simple poles of ζ. This concludes this post.