Title: Introduction to Hamiltonian and Lagrangian Mechanics
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The standard, non-relativistic mathematical physics definition of the Lagrangian L is:

$$L = T - V \tag{1}$$

where T is the kinetic energy and V is the potential. The standard Lagrangian has variable dependence as follows:

$$L = L\left(\mathbf{q}, \dot{\mathbf{q}}, t\right)$$

where $\mathbf{q}$ is the spatial coordinate vector, and $\dot{\mathbf{q}}$ is the time derivative (derivative with respect to a scalar t) of the spatial coordinate vector - i.e. known as the velocity vector in coordinate space. Based on this definition, we define the standard, non-relativistic Hamiltonian H for N phase-space coordinates as:

$$H\left(\mathbf{q}, \mathbf{p}, t\right) = \mathbf{p} \cdot \dot{\mathbf{q}} - L\left(\mathbf{q}, \dot{\mathbf{q}}, t\right) \tag{2}$$

where $\mathbf{p}$ is the momentum vector (either linear or angular), where $H = T + V$ in standard formalism. Note that this implies that:

$$T = \frac{H + L}{2} = \frac{1}{2}\mathbf{p} \cdot \dot{\mathbf{q}}$$

as is the classical definition of kinetic energy, in the case that $\mathbf{p} = m\dot{\mathbf{q}}$ (where m is the scalar mass of the particle or rigid body) for linear momentum or $\mathbf{p} = \underline{\mathbf{I}}\dot{\omega}$ for angular momentum (where $\underline{\mathbf{I}}$ is the moment of inertia rank 2 tensor for the particle or rigid body). The most general definition of this momentum vector is as follows:

$$\begin{aligned} \mathbf{p} &= \frac{\partial L}{\partial \dot{\mathbf{q}}} \\ \dot{\mathbf{p}} &= \frac{\partial L}{\partial \mathbf{q}} \end{aligned} \tag{3}$$

which follows from the mathematical definition of the Lagrangian element dL:

$$dL = \frac{\partial L}{\partial \mathbf{q}} \cdot d\mathbf{q} + \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot d\dot{\mathbf{q}} + \frac{\partial L}{\partial t} dt \tag{4}$$

Both H and L are scalar quantities, but the Hamiltonian exists in phase space rather than in configuration space (where L is defined). To see how these two spaces interact, we take the differential of the Hamiltonian in two different definitions. The first line corresponds directly to the chain rule, while the second, third, and fourth lines directly corresponds to Equation 2 (where the third and fourth line uses Equation 3 on the second line):

$$
\begin{aligned}
dH &= \frac{\partial H}{\partial \mathbf{q}} \cdot d\mathbf{q} + \frac{\partial H}{\partial \mathbf{p}} \cdot d\mathbf{p} + \frac{\partial H}{\partial t} dt \\
&= \mathbf{p} \cdot d\dot{\mathbf{q}} + \dot{\mathbf{q}} \cdot d\mathbf{p} - \left(\frac{\partial L}{\partial \mathbf{q}} \cdot d\mathbf{q} + \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot d\dot{\mathbf{q}} + \frac{\partial L}{\partial t} dt \right) \\
&= \mathbf{p} \cdot d\dot{\mathbf{q}} + \dot{\mathbf{q}} \cdot d\mathbf{p} - \left(\dot{\mathbf{p}} \cdot d\mathbf{q} + \mathbf{p} \cdot d\dot{\mathbf{q}} + \frac{\partial L}{\partial t} dt \right) \\
&= \dot{\mathbf{q}} \cdot d\mathbf{p} - \dot{\mathbf{p}} \cdot d\mathbf{q} - \frac{\partial L}{\partial t} dt
\end{aligned}
\tag{5}
$$

Treating $d\mathbf{p}$, $d\mathbf{q}$, and dt independently and contrasting the three terms in line 1 with the three terms in line 4, we reach Hamilton's Equations:

$$
\begin{aligned}
\dot{\mathbf{q}} &= \frac{\partial H}{\partial \mathbf{p}} \\
\dot{\mathbf{p}} &= -\frac{\partial H}{\partial \mathbf{q}} \\
\frac{\partial L}{\partial t} &= -\frac{\partial H}{\partial t}
\end{aligned}
\tag{6}
$$

These equations are defined as Hamilton's Canonical equations of motions, producing $2N + 1$ coupled first-order ordinary differential equations.

Now that we've established that theory, one may question how I will apply this formalism to other content on this blog. I have not given much coverage of ODEs, instead focusing on integrals and summations. To see the application possibility, we first define an action S:

$$
S = \int_{t_1}^{t_2} \mathrm{d}t \, L\left(t, q_i\left(t\right), \dot{q}_i\left(t\right)\right) \tag{7}
$$

where S is known as the "action", and $q_i\left(t\right)$ is the ith component of the coordinate position q. In Lagrangian mechanics theory, the $q_i\left(t\right)$ function that satisfies the *Euler-Lagrange equation* for the ith component is the one that makes $\delta S \to 0$ - the one that makes the action "stationary". The Euler-Lagrange differential equation is:

$$
\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_i} = 0 \tag{8}
$$

The $\mathbf{q}\left(t\right)$ that satisfies the Euler-Lagrange equation minimizes the functional (for each i), and happens to be the path that nature chooses between two points in configuration space. Now to keep things mathematical rather than physical, we will play with integration by parts in Equation

7:

$$
\begin{aligned}
S &= \int_{q_i(t_1)}^{q_i(t_2)} \mathrm{d}q_i\left(t\right) \, \frac{L\left(q_i^{-1}\left(q_i\left(t\right)\right), q_i\left(t\right), \dot{q}_i\left(t\right)\right)}{\dot{q}_i\left(t\right)} \\
&= \left(q_i\left(t\right) \frac{L\left(q_i^{-1}\left(q_i\left(t\right)\right), q_i\left(t\right), \dot{q}_i\left(t\right)\right)}{\dot{q}_i\left(t\right)}\right)_{q_i(t_1)}^{q_i(t_2)} \\
&\quad - \int_{q_i(t_1)}^{q_i(t_2)} \mathrm{d}q_i\left(t\right) \, q_i\left(t\right) \partial_{q_i(t)} \frac{L\left(q_i^{-1}\left(q_i\left(t\right)\right), q_i\left(t\right), \dot{q}_i\left(t\right)\right)}{\dot{q}_i\left(t\right)} \\
&= q_i\left(t_2\right) \frac{L\left(t_2, q_i\left(t_2\right), \dot{q}_i\left(t_2\right)\right)}{\dot{q}_i\left(t_2\right)} - q_i\left(t_1\right) \frac{L\left(t_1, q_i\left(t_1\right), \dot{q}_i\left(t_1\right)\right)}{\dot{q}_i\left(t_1\right)} \\
&\quad - \int_{q_i(t_1)}^{q_i(t_2)} \mathrm{d}q_i\left(t\right) \, q_i\left(t\right) \left(\frac{\frac{\partial L\left(q_i^{-1}\left(q_i(t)\right), q_i(t), \dot{q}_i(t)\right)}{\partial q_i(t)}}{\dot{q}_i\left(t\right)} - \frac{L\left(q_i^{-1}\left(q_i\left(t\right)\right), q_i\left(t\right), \dot{q}_i\left(t\right)\right)}{\dot{q}_i^3} \ddot{q}_i\right) \\
&= q_i\left(t_2\right) \frac{L\left(t_2, q_i\left(t_2\right), \dot{q}_i\left(t_2\right)\right)}{\dot{q}_i\left(t_2\right)} - q_i\left(t_1\right) \frac{L\left(t_1, q_i\left(t_1\right), \dot{q}_i\left(t_1\right)\right)}{\dot{q}_i\left(t_1\right)} \\
&\quad - \int_{t_1}^{t_2} \mathrm{d}t \, \left(\frac{\partial L}{\partial q_i} q_i\left(t\right) - L\left(t, q_i\left(t\right), \dot{q}_i\left(t\right)\right) \frac{\ddot{q}_i q_i}{\dot{q}_i^2}\right)
\end{aligned}
\tag{9}
$$

Now imagine the case of the non-relativistic particle changing in mass. Then:

$$
S = S\left(t, \mathbf{q}\left(t\right), \dot{\mathbf{q}}\left(t\right), \ddot{\mathbf{q}}\left(t\right)\right)
$$

Now consider what is known as a higher order Lagrangian:

$$
L_2\left(t, q_i\left(t\right), \dot{q}_i\left(t\right), \ddot{q}\left(t\right)\right) = \frac{\partial L}{\partial q_i} q_i - L\left(t, q_i, \dot{q}_i\right) \frac{\ddot{q}_i q_i}{\dot{q}_i^2} \tag{10}
$$

This new (fairly general) Lagrangian L_2, satisfies what is known as a generalized Ostrogradsky's equation:

$$
\frac{\partial L_2}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_2}{\partial \dot{q}_i}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L_2}{\partial \ddot{q}_i}\right) = 0 \tag{11}
$$

Carrying through this calculation:

$$
\begin{aligned}
0 &= \frac{\partial^2 L}{\partial q_i^2} q_i + \frac{\partial L}{\partial q_i} - \frac{\partial L}{\partial q_i} \frac{\ddot{q}_i q_i}{\dot{q}_i^2} - L\left(t, q_i, \dot{q}_i\right) \frac{\ddot{q}_i}{\dot{q}_i^2} \\
&\quad - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial^2 L}{\partial q_i \partial \dot{q}_i} - \frac{\partial L}{\partial \dot{q}} \frac{\ddot{q}_i q_i}{\dot{q}_i^2} + 2L\left(t, q_i, \dot{q}_i\right) \frac{\ddot{q}_i q_i}{\dot{q}_i^3}\right) \\
&\quad + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(-L\left(t, q_i, \dot{q}_i\right) \frac{q_i}{\dot{q}_i^2}\right) \\
0 &= \frac{\partial^2 L}{\partial q_i^2} q_i + \frac{\partial L}{\partial q_i} - \frac{\partial L}{\partial q_i} \frac{\ddot{q}_i q_i}{\dot{q}_i^2} - L\left(t, q_i, \dot{q}_i\right) \frac{\ddot{q}_i}{\dot{q}_i^2} \\
&\quad - \left(\frac{\mathrm{d}\left(\frac{\partial^2 L}{\partial q_i \partial \dot{q}_i}\right)}{\mathrm{d}t} q_i + \frac{\partial^2 L}{\partial q_i \partial \dot{q}_i} \dot{q}_i - \frac{\mathrm{d}\left(\frac{\partial L}{\partial \dot{q}}\right)}{\mathrm{d}t} \frac{\ddot{q}_i q_i}{\dot{q}_i^2} - \frac{\partial L}{\partial \dot{q}} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\ddot{q}_i q_i}{\dot{q}_i^2} + 2 \frac{\mathrm{d}L\left(t, q_i, \dot{q}_i\right)}{\mathrm{d}t} \frac{\ddot{q}_i q_i}{\dot{q}_i^3} + 2L\left(t, q_i, \dot{q}_i\right) \frac{\mathrm{d}}{\mathrm{d}t} \frac{\ddot{q}_i q_i}{\dot{q}_i^3}\right) \\
&\quad - \frac{\mathrm{d}^2 L\left(t, q_i, \dot{q}_i\right)}{\mathrm{d}t^2} \frac{q_i}{\dot{q}_i^2} - 2 \frac{\mathrm{d}L\left(t, q_i, \dot{q}_i\right)}{\mathrm{d}t} \frac{\mathrm{d}}{\mathrm{d}t} \frac{q_i}{\dot{q}_i^2} - L\left(t, q_i, \dot{q}_i\right) \frac{\mathrm{d}^2}{\mathrm{d}t^2} \frac{q_i}{\dot{q}_i^2}
\end{aligned}
\tag{12}
$$

In order, we specify the derivatives of the ratio of q terms:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\ddot{q}_i q_i}{\dot{q}_i^2} = \frac{\dddot{q}_i q_i}{\dot{q}_i^2} + \frac{\ddot{q}_i}{\dot{q}} - 2\frac{\ddot{q}_i^2 q_i}{\dot{q}_i^3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\ddot{q}_i q_i}{\dot{q}_i^3} = \frac{\dddot{q}_i q_i}{\dot{q}_i^3} + \frac{\ddot{q}_i}{\dot{q}_i^2} - 3\frac{\ddot{q}_i^2 q_i}{\dot{q}_i^4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{q_i}{\dot{q}_i^2} = \frac{1}{\dot{q}_i} - 2\frac{\ddot{q}_i q_i}{\dot{q}_i^3}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{q_i}{\dot{q}_i^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\dot{q}_i} - 2\frac{\ddot{q}_i q_i}{\dot{q}_i^3}\right) = -3\frac{\ddot{q}}{\dot{q}_i^2} - 2\frac{\dddot{q}_i q_i}{\dot{q}_i^3} + 6\frac{\ddot{q}_i^2 q_i}{\dot{q}_i^4}$$

Substituting these derivatives into Equation 12 and eliminating terms, we get:

$$\begin{aligned}
0 = &\frac{\partial^2 L}{\partial q_i^2}q_i + \frac{\partial L}{\partial q_i} - \frac{\partial L}{\partial q_i}\frac{\ddot{q}_i q_i}{\dot{q}_i^2} - L\left(t, q_i, \dot{q}_i\right)\frac{\ddot{q}_i}{\dot{q}_i^2} - \frac{\mathrm{d}\left(\frac{\partial^2 L}{\partial q_i \partial \dot{q}_i}\right)}{\mathrm{d}t}q_i - \frac{\partial^2 L}{\partial q_i \partial \dot{q}_i}\dot{q}_i + \frac{\mathrm{d}\left(\frac{\partial L}{\partial \dot{q}}\right)}{\mathrm{d}t}\frac{\ddot{q}_i q_i}{\dot{q}_i^2} \\
&- \left(-\frac{\partial L}{\partial \dot{q}_i}\left(\frac{\dddot{q}_i q_i}{\dot{q}_i^2} + \frac{\ddot{q}_i}{\dot{q}} - 2\frac{\ddot{q}_i^2 q_i}{\dot{q}_i^3}\right) + 2\frac{\mathrm{d}L\left(t, q_i, \dot{q}_i\right)}{\mathrm{d}t}\frac{\ddot{q}_i q_i}{\dot{q}_i^3} + 2L\left(t, q_i, \dot{q}_i\right)\left(\frac{\dddot{q}_i q_i}{\dot{q}_i^3} + \frac{\ddot{q}_i}{\dot{q}_i^2} - 3\frac{\ddot{q}_i^2 q_i}{\dot{q}_i^4}\right)\right) \\
&- \frac{\mathrm{d}^2 L\left(t, q_i, \dot{q}_i\right)}{\mathrm{d}t^2}\frac{q_i}{\dot{q}_i^2} - 2\frac{\mathrm{d}L\left(t, q_i, \dot{q}_i\right)}{\mathrm{d}t}\left(\frac{1}{\dot{q}_i} - 2\frac{\ddot{q}_i q_i}{\dot{q}_i^3}\right) - L\left(t, q_i, \dot{q}_i\right)\left(-3\frac{\ddot{q}}{\dot{q}_i^2} - 2\frac{\dddot{q}_i q_i}{\dot{q}_i^3} + 6\frac{\ddot{q}_i^2 q_i}{\dot{q}_i^4}\right)
\end{aligned} \tag{13}$$

Now, to check this formula, we will take the specific formula where m is a constant, and $L = T = \frac{1}{2}m\dot{q}_i^2$:

$$\begin{aligned}
0 = &-\frac{1}{2}m\ddot{q}_i + m\frac{\ddot{q}_i^2 q_i}{\dot{q}_i^2} \\
&- \left(-m\frac{\dddot{q}_i q_i}{\dot{q}_i} - m\ddot{q}_i + 2m\frac{\ddot{q}_i^2 q_i}{\dot{q}_i^2} + 2m\frac{\ddot{q}_i^2 q_i}{\dot{q}_i^2} + m\left(\frac{\dddot{q}_i q_i}{\dot{q}_i} + \ddot{q}_i - 3\frac{\ddot{q}_i^2 q_i}{\dot{q}_i^2}\right)\right) \\
&- m\left(\dot{q}_i \dddot{q}_i + \ddot{q}_i^2\right)\frac{q_i}{\dot{q}_i^2} - 2m\dot{q}_i\ddot{q}_i\left(\frac{1}{\dot{q}_i} - 2\frac{\ddot{q}_i q_i}{\dot{q}_i^3}\right) - \frac{1}{2}m\dot{q}_i^2\left(-3\frac{\ddot{q}}{\dot{q}_i^2} - 2\frac{\dddot{q}_i q_i}{\dot{q}_i^3} + 6\frac{\ddot{q}_i^2 q_i}{\dot{q}_i^4}\right) \\
0 = &m\ddot{q}_i\left(-\frac{1}{2} + 1 + 1 - 1 - 2 + \frac{3}{2}\right) + m\frac{\ddot{q}_i^2 q_i}{\dot{q}_i^2}\left(1 - 2 - 2 + 3 - 1 + 2 - 3\right) \\
&+ \frac{\dddot{q}_i q_i}{\dot{q}_i}\left(-1 + 1 - 1 + 1\right) \\
0 = &0
\end{aligned} \tag{14}$$

as expected. So specifically, what have we done? We have found the differential equation which, when solved for configuration space paths, makes the L_2 integral in Equation 9 stationary. But there is an idea that I feel is worth mentioning here, even though it is *too ambitious*. Consider constraining the solution further - we can also specify the conventional, 1st order Euler-Lagrange equation to the top line of Equation 9, such that in addition to optimizing for the paths that lead to the L_2 integral being stationary, we also optimize for the L integral being stationary. However, this action would likely lead to a differential equation with no solution. Nonetheless, in an attempt to understand the curious nature of the L_2 integral, I did some internet searching and happened upon some interesting findings. First off, L_2 is "affine" because of the following Hessian matrix identity:

$$\frac{\partial^2 L_2}{\partial \ddot{q}^2} = 0$$

This concept of L_2 being linear in $\ddot{q}$ means that it is "maximally" degenerate, and the Hessian has a vanishing determinant. This realization has two important implications:

- The phase space coordinates are constrained primarily - they have a relation that eliminates a degree of freedom in phase space *without* considering the equation of motion - it follows directly from the definition of momentum. To explain this further, we will return to the definition of momenta in Equation 3. The phase space behaviour of the second order dynamical system is ordinarily dependent on four phase space coordinates:

  $$(q, \dot{q}, p_0, p_1)$$

  where:

  $$p_0 = \frac{\partial L_2}{\partial \dot{q}}$$
  $$p_1 = \frac{\partial L_2}{\partial \ddot{q}}$$

  However, in this case, because p_1 is independent of $\ddot{q}$, so we find that there exists a constraining relationship between p_1, q, and $\dot{q}$ that removes a degree of freedom in the system:

  $$p_1 + \frac{q}{\dot{q}^2} L = 0$$

  For this reason, this system avoids something known as the "Ostrogradsky instability" resulting from the 2nd order Lagrangian differential equation. This concept causes the solutions to the equations of motion to become unstable (unbounded above and below - thus undesirable physically) under certain conditions if no degeneracy exists in L_2 that removes a degree of freedom from a primary (definition) constraint of the phase space variables.

- The concept of degeneracy means that there should be a relevant concept known in mathematical physics as gauge theory. From my own grad school training, the most familiar example of this idea is the use of the Lorentz Gauge to convert Maxwell's equations to wave equation form in electromagnetism. But this idea is quite general to a wide variety of degenerate Lagrangian systems. To understand this idea, define an arbitrary function F defined as follows:

  $$\begin{aligned} \tilde{L}_2 &= L_2 + \frac{\mathrm{d}F(t, q, \dot{q})}{\mathrm{d}t} \\ \text{where } \frac{\mathrm{d}F}{\mathrm{d}t} &= \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial \dot{q}}\ddot{q} \end{aligned} \tag{15}$$

  where if we want to maintain the same form of differential equation as L_2 (with no change to the family of solutions of the original Lagrangian), we must impose constraints on F derived from Equation 15:

  $$\begin{aligned} \frac{\partial F}{\partial \dot{q}} &= -\frac{q}{\dot{q}^2} L \\ \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q}\dot{q} &= q\frac{\partial L}{\partial q} \end{aligned} \tag{16}$$

Now, these equations, if satisfied, produce a function F (given a 1st order Lagrangian L) whose time derivative can be freely added to L_2 without changing the form of the second order Lagrangian. This transformation is possible due to the affine nature of L_2, and is called the "gauge invariance" of L_2.

Now that I have summarized some of the theory related to this special L_2 that has been derived, it's now time to progress to compare the solutions attained when optimizing in terms of L (Equation 7) and in terms of L_2 (Equation 9). To do this, consider one of the simplest Lagrangian systems in physics - the harmonic oscillator:

$$\begin{aligned} L &= \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2 \\ L_2 &= -kq^2 - \left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2\right)\frac{\ddot{q}q}{\dot{q}^2} \end{aligned} \tag{17}$$

We have assumed $k = 1 = m$ for simple investigation. To write out the ODEs explicitly:

$$\begin{aligned} \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) &= 0 \\ \implies \ddot{q} + q &= 0 \end{aligned} \tag{18}$$

The ODE for L_2 is trickier, and involves the following calculations:

$$\frac{\partial L_2}{\partial q} = -2q - \frac{1}{2}\ddot{q} + \frac{3}{2}\frac{\ddot{q}q^2}{\dot{q}^2}$$

$$\frac{\partial L_2}{\partial \dot{q}} = -\frac{\ddot{q}q^3}{\dot{q}^3} \implies \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_2}{\partial \dot{q}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{\ddot{q}q^3}{\dot{q}^3}\right) = \left(-\frac{\dddot{q}\,q^3}{\dot{q}^3} - 3\frac{\ddot{q}q^2}{\dot{q}^2} + 3\frac{\ddot{q}^2q^3}{\dot{q}^4}\right)$$

$$\frac{\partial L_2}{\partial \ddot{q}} = \frac{1}{2}\frac{q^3}{\dot{q}^2} - \frac{1}{2}q \implies \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial L_2}{\partial \ddot{q}} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{1}{2}\frac{q^3}{\dot{q}^2} - \frac{1}{2}q\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3}{2}\frac{q^2}{\dot{q}} - \frac{\ddot{q}q^3}{\dot{q}^3}\right) - \frac{1}{2}\ddot{q}\right)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial L_2}{\partial \ddot{q}} = \left(\left(3q - \frac{3}{2}\frac{\ddot{q}q^2}{\dot{q}^2} - \frac{\dddot{q}\,q^3}{\dot{q}^3} - 3\frac{\ddot{q}q^2}{\dot{q}^2} + 3\frac{\ddot{q}^2q^3}{\dot{q}^4}\right) - \frac{1}{2}\ddot{q}\right)$$

The higher-order Lagrangian ODE is:

$$\begin{aligned} &\frac{\partial L_2}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_2}{\partial \dot{q}}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial L_2}{\partial \ddot{q}} = 0 \\ &\implies -2q - \frac{1}{2}\ddot{q} + \frac{3}{2}\frac{\ddot{q}q^2}{\dot{q}^2} - \left(-\frac{\dddot{q}\,q^3}{\dot{q}^3} - 3\frac{\ddot{q}q^2}{\dot{q}^2} + 3\frac{\ddot{q}^2q^3}{\dot{q}^4}\right) \\ &\qquad\quad + \left(\left(3q - \frac{3}{2}\frac{\ddot{q}q^2}{\dot{q}^2} - \frac{\dddot{q}\,q^3}{\dot{q}^3} - 3\frac{\ddot{q}q^2}{\dot{q}^2} + 3\frac{\ddot{q}^2q^3}{\dot{q}^4}\right) - \frac{1}{2}\ddot{q}\right) = 0 \\ &\implies \ddot{q} - q = 0 \end{aligned} \tag{19}$$

The two ODEs in Equation 18 and Equation 19 are solved as follows:

$$\begin{aligned} \ddot{q}_1 + q_1 = 0 &\implies q_1(t) = A\sin(t) + B\cos(t) \\ \ddot{q}_2 - q_2 = 0 &\implies q_2(t) = Ce^{t} + De^{-t} \end{aligned} \tag{20}$$

The interesting part comes in the resultant actions:

$$
\begin{aligned}
S_1 &= \int_{t_1}^{t_2} \mathrm{d}t \, L\left(t, q_1, \dot{q}_1\right) \\
&= \frac{1}{2}\int_{t_1}^{t_2} \mathrm{d}t \, \left(\dot{q}_1^2 - q_1^2\right) \\
&= \frac{1}{2}\int_{t_1}^{t_2} \mathrm{d}t \, \Big(\left(A\cos\left(t\right) - B\sin\left(t\right)\right)^2 - \left(A\sin\left(t\right) + B\cos\left(t\right)\right)^2\Big) \\
&= \frac{1}{2}\int_{t_1}^{t_2} \mathrm{d}t \, \left(A^2\left(\cos^2\left(t\right) - \sin^2\left(t\right)\right) + B^2\left(\sin^2\left(t\right) - \cos^2\left(t\right)\right) - 4AB\cos\left(t\right)\sin\left(t\right)\right) \\
&= \frac{\left(A^2 - B^2\right)}{2}\int_{t_1}^{t_2} \mathrm{d}t \, \left(\cos\left(2t\right)\right) - AB\int_{t_1}^{t_2} \mathrm{d}t \, \left(\sin\left(2t\right)\right) \\
&= \frac{\left(A^2 - B^2\right)}{4}\left(\sin\left(2t_2\right) - \sin\left(2t_1\right)\right) + \frac{AB}{2}\left(\cos\left(2t_1\right) - \cos\left(2t_2\right)\right)
\end{aligned}
\tag{21}
$$

Consider initial conditions:

$$q\left(t_1\right) = q\left(0\right) = 1 \,\text{and}\, \dot{q}\left(t_1\right) = \dot{q}\left(0\right) = 1$$

where $t_1 = 0$ and where $t_2 = 1$. This implies that $q_1\left(t\right)$ is:

$$q_1\left(t\right) = \sin\left(t\right) + \cos\left(t\right)$$

$$q_2\left(t\right) = e^t$$

Thus, the actions are:

$$
\begin{aligned}
S_1 &= \frac{1}{2}\left(1 - \cos\left(2\right)\right) \\
S_2 &= \int_0^1 \mathrm{d}t \, \left(q_2\frac{\partial L\left(t, q_2, \dot{q}_2\right)}{\partial q_2} - L\left(t, q_2, \dot{q}_2\right)\frac{\ddot{q}_2 q_2}{\dot{q}_2^2}\right) \\
S_2 &= \int_0^1 \mathrm{d}t \, \left(-q_2^2 - \frac{1}{2}\left(\dot{q}_2^2 - q_2^2\right)\right) \\
&= -\frac{1}{2}\int_0^1 \mathrm{d}t \, \left(\dot{q}_2^2 + q_2^2\right) \\
&= -\int_0^1 \mathrm{d}t \, \left(e^{2t}\right) \\
&= -\frac{1}{2}\left(e^2 - 1\right) \\
\implies \tilde{S}_1 &= q_2\left(t_2\right)\frac{L\left(t_2, q\left(t_2\right), \dot{q}_2\left(t_2\right)\right)}{\dot{q}_2\left(t_2\right)} - q_2\left(t_1\right)\frac{L\left(t_1, q\left(t_1\right), \dot{q}_2\left(t_1\right)\right)}{\dot{q}_2\left(t_1\right)} - S_2 \\
&= \frac{e^2 - 1}{2}
\end{aligned}
\tag{22}
$$

And, as we can see the $\tilde{S}_1$ action is substantially greater than S_1, as least for these initial conditions. Furthermore, it is interesting if we can find initial conditions for which $\tilde{S}_1 < S_1$:

$$t_1 = 0, t_2 = 1, q\left(t_1\right) = 1, \dot{q}\left(t_1\right) = 0$$

So:

$$q_1(t) = \cos(t)$$

$$q_2(t) = \frac{1}{2}\left(e^t + e^{-t}\right)$$

From the general formula in Equation 21, we find that S_1 general formula. To find the general formula for S_2:

$$\begin{aligned} S_2 &= \int_{t_1}^{t_2} \mathrm{d}t \left(q_2 \frac{\partial L(t, q_2, \dot{q}_2)}{\partial q_2} - L(t, q_2, \dot{q}_2) \frac{\ddot{q}_2 q_2}{\dot{q}_2^2} \right) \\ &= \int_{t_1}^{t_2} \mathrm{d}t \left(De^{-t} \frac{\left(Ce^t + De^{-t}\right)^2}{\left(Ce^t - De^{-t}\right)^2} - \left(Ce^t + De^{-t}\right)^2 \right) \end{aligned} \tag{23}$$

To now rewrite A, B, C, and D in terms of $q(t_1)$ and $\dot{q}(t_1)$ where $t_1 = 0$.

$$B = q(0), A = \dot{q}(0)$$

$$D = \frac{q(0) - \dot{q}(0)}{2}, C = \frac{q(0) + \dot{q}(0)}{2}$$

Plotting these two expressions as 3D surfaces in terms of the two varying quantities - $q(0)$ and $\dot{q}(0)$ - we attain the Figure 1. We vary the initial conditions only between 0 and 1 on the real number

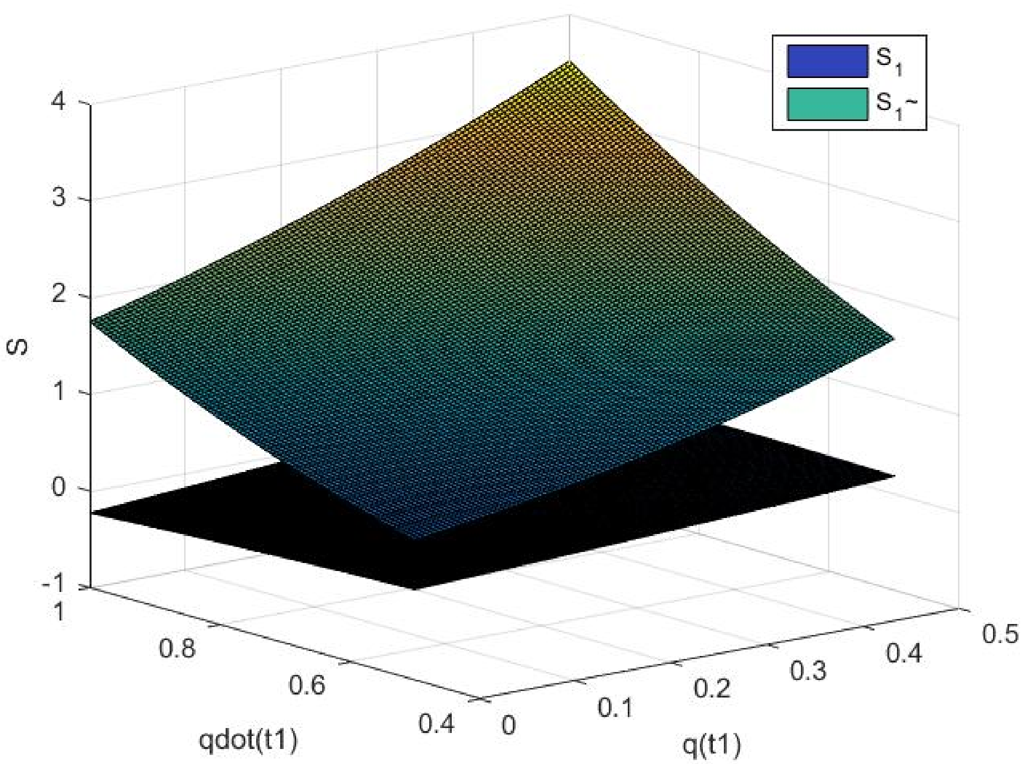

Figure 1: The lower surface is S_1 while the upper surface is $\tilde{S}_1$.

line, where:

$$S_1 = \int_{t_1}^{t_2} \mathrm{d}t \, L\left(t, q_1, \dot{q}_1\right)$$

$$\tilde{S}_1 = \frac{q_2\left(t_2\right)}{\dot{q}_2\left(t_2\right)} L\left(t_2, q_2, \dot{q}_2\right) - \frac{q_2\left(t_1\right)}{\dot{q}_2\left(t_1\right)} L\left(t_1, q_2, \dot{q}_2\right) - \int_{t_1}^{t_2} \mathrm{d}t \, \left(q_2 \frac{\partial L}{\partial_{q_2}} - L\left(t, q_2, \dot{q}_2\right) \frac{\ddot{q}_2 q_2}{\dot{q}_2^2}\right)$$

$$q_1\left(t\right) = A \sin\left(t\right) + B \cos\left(t\right)$$

$$q_2\left(t\right) = Ce^{t} + De^{-t}$$

As can be seen in Figure 1, there is no domain over which $\tilde{S}_1 > S_1$. These results are limited to the domain plotted and the harmonic oscillator Lagrangian for this example, but given the implicit connection between L and L_2, I would tend towards believing that it represents a general trend, because L purely optimizes S_1, whereas L_2 only optimizes a term in $\tilde{S}_1$ - the two $\tilde{S}_1$ terms from the integration by parts are not considered in the optimization. Finally, we will write the corresponding phase space Hamilton's first order equations for this second order Lagrangian system, to be solved in the next post:

$$\begin{aligned} H_2\left(\mathbf{q}, \mathbf{v}, \mathbf{p}, \pi\right) &= \mathbf{p} \cdot \mathbf{v} + \pi \cdot \mathbf{a}\left(\mathbf{q}, \mathbf{v}, \pi\right) - L_2\left(\mathbf{q}, \mathbf{v}, \mathbf{a}\left(\mathbf{q}, \mathbf{v}, \pi\right)\right) \\ \dot{\mathbf{q}} &= \frac{\partial H_2}{\partial \mathbf{p}} = \mathbf{v} \\ \dot{\mathbf{v}} &= \frac{\partial H_2}{\partial \pi} = \mathbf{a} \\ \dot{\mathbf{p}} &= -\frac{\partial H_2}{\partial \mathbf{q}} \\ \dot{\pi} &= -\frac{\partial H_2}{\partial \mathbf{v}} \\ \mathbf{p} &= \frac{\partial L_2}{\partial \mathbf{v}} - \dot{\pi} \\ \pi &= \frac{\partial L_2}{\partial \dot{\mathbf{v}}} \end{aligned} \tag{24}$$

where the first equation corresponds to the canonical momentum (without the constraint terms, and the last two equations are constraint equations (for the degenerate case) rather than simply a canonical momenta definition, constraining the phase space solution to ensure the solution is bounded from above and below. We will take on this set of Equations in the next post. This concludes this blog post.