Title: Infinite Summations Involving the Harmonic Number
Author: Josh Myers

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So I went to begin writing my post about De-Yin Zheng's 2007 paper explaining the method of obtaining infinite series involving powers of the harmonic number. This post was to be an extension of the logarithmic integrals post where it was found that the integration of the logarithmic function was only in terms of an infinite series of powers of the Harmonic Numbers divided by powers of the series index. Zheng's paper provides a very powerful method of expressing these series in closed form in terms of the Riemann Zeta function via equating powers of a multivariate series expansion with a series expansion involving the described Harmonic Numbers. However, as I dug into this paper, I became curious about something else. The paper gives the following identity without proof:

$$\prod_{k=1}^{n}\left(1+\frac{x}{k}\right)=1+H_n x+\left(H_n^2-H_n^{(2)}\right)\frac{x^2}{2!}+\left(H_n^3-3H_nH_n^{(2)}+2H_n^{(3)}\right)\frac{x^3}{3!}+\dots \tag{1}$$

This series expansion is not explicitly expressed in terms of an integer index variable as a power series, but the coefficients look remarkably like the Harmonic Number explicit expressions for the Stirling Numbers of the First Kind, found in another post on this blog. If the connection were genuine and not just a pattern that shows up for the first finite number of terms, then the general series expression would be:

$$\prod_{k=1}^{n}\left(1+\frac{x}{k}\right)=\sum_{k=0}^{n}\left(\frac{\left[{n+1 \atop k+1}\right]}{n!}x^k\right) \tag{2}$$

If this series generalization is true, then there are some interesting implications. First, consider another similar identity (used in other posts on this site) relating powers of the logarithm to a series of Stirling Numbers of the First Kind:

$$\frac{\left(-\ln\left(1-x\right)\right)^k}{k!}=\sum_{n=k}^{\infty}\left(\frac{\left[{n \atop k}\right]}{n!}x^n\right) \tag{3}$$

The main problem with trying to compare the series' in Equation 2 and Equation 3 is that the power series of Equation 2 is in terms of the lower argument of the Stirling Numbers while Equation 3 power series is in terms of the upper argument of the Stirling Numbers. To simplify this, let $x=1$ in Equation 2. Then sum Equation 2 in terms of an infinite series in terms of n and sum Equation

3 in terms of an infinite series in k. First, multiplying Equation 2 by $\frac{x^{n+1}}{n+1}$ and summing:

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\frac{x^{n+1}}{n+1}\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)\right) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\frac{\left[{n+1 \atop k+1}\right]}{(n+1)!}x^{n+1}\right) \\ &= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\left(\frac{\left[{n+1 \atop k+1}\right]}{(n+1)!}x^{n+1}\right)\end{aligned} \tag{4}$$

Now indexing up Equation 3 in k and then summing, noting that the left hand side is simply an exponential Maclaurin series in the logarithm:

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{\left(\ln\left(\frac{1}{1-x}\right)\right)^{k+1}}{(k+1)!} = \frac{x}{1-x} &= \sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty}\left(\frac{\left[{n \atop k+1}\right]}{n!}x^{n}\right) \\ &= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\left(\frac{\left[{n+1 \atop k+1}\right]}{(n+1)!}x^{n+1}\right)\end{aligned} \tag{5}$$

where the second line is simply an up-index in n. Now we recognize that the right sides of Equation 4 and Equation 5 match perfectly; thus, the left sides can be set equal. Then, expand $\frac{x}{1-x}$ in terms of the geometric series, and then set the coefficients of the powers of x equal to one another. This gives the simple result:

$$\frac{\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)}{n+1} = 1 \tag{6}$$

Though this identity seems somewhat unexpected, it is actually a simple consequence of a telescoping product (rather than the series analog), where the numerator of each term cancels with the denominator of the next term, until only the last numerator and the first denominator are left. Since the first denominator is always 1 and the last numerator is always $n+1$, this means that dividing this result by $n+1$ trivially gives the right side result.

So what have we learned from this calculation? We obtained a trivially true result from conjecture that Equation 2 is true. Does this argument necessarily mean that Equation 2 is true? From only this argument, it seems not to be enough - after all, it seems fully possible that we attained a true result from untrue assumptions. But there is something more to think about. If we take Equation 3 to be true, then all that is required is for Equation 4 to be related to Equation 5 in the way necessary for the Equation 6 to be true. But of course, since we have specified the summation behaviour of Equation 5, then it means that Equation 4 has the same behaviour. This "same" behaviour implies the conjectured result (Equation 2), and with this argument, it seems to be enough to elevate the result from conjecture to identity.

Next will be my coverage of De-Yin Zheng's 2007 paper. This concludes this blog post.