Title: General Properties of Functions on the Complex Plane
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Consider the common general function $\eta(s)$ where $s = \sigma + it$ with the following property:

$$\eta(\bar{s}) = \overline{\eta(s)} \tag{1}$$

Now consider the absolute value of the general function:

$$|\eta(s)| = \sqrt{\eta(s)\,\overline{\eta(s)}} = \sqrt{\eta(s)\,\eta(\bar{s})}$$

Taking the derivatives of this expression with respect to σ and t:

$$\begin{aligned}
\partial_\sigma\left(|\eta(\sigma+it)|\right) &= \frac{1}{2}\sqrt{\frac{\eta(\bar{s})}{\eta(s)}}\eta'(s) + \frac{1}{2}\sqrt{\frac{\eta(s)}{\eta(\bar{s})}}\eta'(\bar{s}) \\
\partial_t\left(|\eta(\sigma+it)|\right) &= \frac{i}{2}\sqrt{\frac{\eta(\bar{s})}{\eta(s)}}\eta'(s) - \frac{i}{2}\sqrt{\frac{\eta(s)}{\eta(\bar{s})}}\eta'(\bar{s}) \\
\implies \partial_\sigma\left(|\eta(\sigma+it)|\right) + i\partial_t\left(|\eta(\sigma+it)|\right) &= \sqrt{\frac{\eta(s)}{\eta(\bar{s})}}\eta'(\bar{s}) \\
\implies \partial_\sigma\left(|\eta(\sigma+it)|\right) - i\partial_t\left(|\eta(\sigma+it)|\right) &= \sqrt{\frac{\eta(\bar{s})}{\eta(s)}}\eta'(s)
\end{aligned} \tag{2}$$

where we recognize the derivatives from the first and second lines take very similar forms to what is known as the Wirtinger derivatives in complex analysis, introduced in 1927 by Wilhelm Wirtinger. Dividing the fourth line by the third line in Equation 2, we obtain a nice result:

$$\begin{aligned}
&\frac{\partial_\sigma\left(|\eta(\sigma+it)|\right) - i\partial_t\left(|\eta(\sigma+it)|\right)}{\partial_\sigma\left(|\eta(\sigma+it)|\right) + i\partial_t\left(|\eta(\sigma+it)|\right)} = \frac{\eta(\bar{s})}{\eta(s)}\frac{\eta'(s)}{\eta'(\bar{s})} \\
\implies &\frac{\eta'(\bar{s})}{\eta(\bar{s})}\left(\partial_\sigma\left(|\eta(\sigma+it)|\right) - i\partial_t\left(|\eta(\sigma+it)|\right)\right) = \left(\partial_\sigma\left(|\eta(\sigma+it)|\right) + i\partial_t\left(|\eta(\sigma+it)|\right)\right)\frac{\eta'(s)}{\eta(s)}
\end{aligned} \tag{3}$$

where we can see in the second line of Equation 3 that the left side and right side terms are complex conjugates of each other. The presence of the equality means that both sides must thus be real. This realization, while significant, is not entirely surprising - after all, $|\eta(s)|$ is real by the definition of the norm. To now zoom in on the implications of this realization, we secondly realize that the right and left sides of the first line of Equation 3 both have an absolute value of 1. This means that we can without loss of generality specify:

$$\chi(\sigma,t) = e^{-2i\theta(\sigma,t)} = \frac{\partial_\sigma\left(|\eta(\sigma+it)|\right) - i\partial_t\left(|\eta(\sigma+it)|\right)}{\partial_\sigma\left(|\eta(\sigma+it)|\right) + i\partial_t\left(|\eta(\sigma+it)|\right)} = \frac{\eta'(\sigma+it)}{\eta(\sigma+it)}\frac{\eta(\sigma-it)}{\eta'(\sigma-it)}$$

where $\theta(\sigma,t)$ is a real valued function and $\chi(\sigma,t)$ is complex valued, such that we arrive at an Equation that is similar in form to the derivation of the Hardy Z function for the Riemann Zeta function:

$$e^{i\theta(\sigma,t)}\frac{\eta'(\sigma+it)}{\eta(\sigma+it)} = e^{-i\theta(\sigma,t)}\frac{\eta'(\sigma-it)}{\eta(\sigma-it)} \tag{4}$$

such that one can define the following Z function:

$$\begin{aligned} Z(\sigma,t) &= e^{i\theta(\sigma,t)}\frac{\eta'(\sigma+it)}{\eta(\sigma+it)} \\ &= e^{i\theta(\sigma,t)}\partial_\sigma \ln(\eta(\sigma+it)) \end{aligned} \tag{5}$$

We have been quite general in our derivation thus far, and it becomes clear that such a $theta(\sigma+it)$ and strictly real-valued $Z(\sigma,t)$ exist for any function $\eta(\sigma+it)$ for which Equation 1 holds. Thus, there is always a real-valued function Z that maintains the magnitude of the ratio of the complex function's derivative to the value of the complex function, such that $|Z(\sigma,t)| = \left|\frac{\eta'(\sigma+it)}{\eta(\sigma+it)}\right|$. The question then arises about the complexity of $\theta(\sigma,t)$. Of course, a noteworthy point is that an expression of this nature on its own is not profound - after all, the complex function on its own is always expressible as a real-valued function for a general real valued $\alpha(\sigma,t)$:

$$|\eta(\sigma+it)| = e^{i\alpha(\sigma,it)}\eta(\sigma+it) \tag{6}$$

where:

$$\alpha(\sigma,it) = -\arctan\left(\frac{\mathrm{Im}(\eta(\sigma+it))}{\mathrm{Re}(\eta(\sigma+it))}\right)$$

where it immediately follows that a real-valued $Z_2(\sigma,t)$ can be specified further as:

$$Z_2(\sigma,t) = |\eta(\sigma+it)|\,Z(\sigma,t) = e^{i(\theta(\sigma,t)+\alpha(\sigma,t))}\eta'(\sigma+it) = e^{i\theta_2(\sigma,t)}\eta'(\sigma+it) \tag{7}$$

which, on the surface, is not profound, given that Equation 6 holds, and $\eta'(\sigma+it)$ is a general complex valued function just as is $\eta(\sigma+it)$. To investigate this idea further, we note that from Equation 6, we have:

$$\begin{aligned} \theta_2(\sigma,t) &= -\arctan\left(\frac{\mathrm{Im}(\eta'(\sigma+it))}{\mathrm{Re}(\eta'(\sigma+it))}\right) \\ \text{Also, } \theta_2(\sigma,t) = \theta(\sigma,t)+\alpha(\sigma,t) &= -\frac{1}{2}\mathrm{Im}(\ln(\chi(\sigma,t))) - \arctan\left(\frac{\mathrm{Im}(\eta(\sigma+it))}{\mathrm{Re}(\eta(\sigma+it))}\right) \\ \implies \arctan\left(\frac{\mathrm{Im}(\eta'(\sigma+it))}{\mathrm{Re}(\eta'(\sigma+it))}\right) &= \frac{1}{2}\mathrm{Im}(\ln(\chi(\sigma,t))) + \arctan\left(\frac{\mathrm{Im}(\eta(\sigma+it))}{\mathrm{Re}(\eta(\sigma+it))}\right) \\ \theta(\sigma,t) &= -\arctan\left(\frac{\mathrm{Im}(\eta'(\sigma+it))}{\mathrm{Re}(\eta'(\sigma+it))}\right) + \arctan\left(\frac{\mathrm{Im}(\eta(\sigma+it))}{\mathrm{Re}(\eta(\sigma+it))}\right) \end{aligned} \tag{8}$$

Specifying $\chi(\sigma+it) = \frac{\eta'(\sigma+it)}{\eta(\sigma+it)}\frac{\eta(\sigma-it)}{\eta'(\sigma-it)}$, I have confirmed this relation numerically. In my next post, I will proceed to evaluate Equation 5 for the special case of the Riemann Zeta function, which obeys Equation 1. This concludes this blog post.