Title: The Superfactorial and the Barnes G Function
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Define the superfactorial sf as the following:

$$sf\left(n\right) = \prod_{k}^{n} k! \tag{1}$$

This math object came up when I tried applying the geometric mean theory - the one I examined with the Riemann Hypothesis in my last post - to the exponential series with real number x:

$$e^{x} = \lim_{n\to\infty} \sum_{k=0}^{n-1} \left(\frac{x^{k}}{k!}\right) \tag{2}$$

Recall the geometric mean-arithmetic mean inequality:

$$\sqrt[n]{\prod_{i=1}^{n} x_i} \leq \frac{\sum_{i=1}^{n} x_i}{n} \tag{3}$$

where equality holds only when all x_i are the same real value, otherwise geometric mean is always less than arithmetic mean. Each x_i must be greater than 0. Applying this theorem to Equation 2, we get:

$$\begin{aligned} e^{x} &= \lim_{n\to\infty} \left(n\frac{\sum_{k=0}^{n-1} \frac{x^{k}}{k!}}{n}\right) \\ &\geq \lim_{n\to\infty} n \sqrt[n]{\prod_{k=0}^{n-1} \frac{x^{k}}{k!}} \\ &= \lim_{n\to\infty} n x^{\frac{n-1}{2}} \sqrt[n]{\frac{1}{sf\left(n-1\right)}} \end{aligned} \tag{4}$$

where we see the superfactorial of Equation 1 comes in. To evaluate this superfactorial, we will derive a useful asymptotic series form (as $n \to \infty$) from a known method I found online (this part is not an original study) that uses machinery known as the Euler-Maclaurin summation formula whose proof I will cover in another post. However, I will present the connection between the superfactorial and the Barnes G function. Define $G\left(n\right)$:

$$G\left(n\right) = \frac{\left(\Gamma\left(n\right)\right)^{n-1}}{H\left(n-1\right)} \tag{5}$$

where $H(n)$ is the hyperfactorial function, defined as:

$$H(n) = \prod_{k=1}^{n} k^k \tag{6}$$

This hyperfactorial definition is closely related to the superfactorial, since the superfactorial (Equation 1) can be re-written as:

$$\begin{aligned} sf(n) = 1^n 2^{n-1} 3^{n-2} \cdots (n-1)^2 n &= \prod_{k=1}^{n} k^{n-k+1} \\ &= \frac{((n+1)!)^{n+1}}{\prod_{k=1}^{n+1} k^k} \end{aligned} \tag{7}$$

This superfactorial formula on the integers is related to the Barnes G function by:

$$sf(n) = G(n+2) \tag{8}$$

With this content defined, the task in finding an asymptotic relation for $sf(n)$ is to find an asymptotic relation for the hyperfactorial, since the common Stirling Formula is known for $\Gamma(n+2)$. To show the hyperfactorial result, from a proof I found on math stack exchange, we do the following to Equation 6:

$$\ln(H(n)) = \sum_{k=1}^{n} k \ln(k) \tag{9}$$

Then, by the Euler-Maclaurin summation theorem (to be proven in my next post), we get the following approximation:

$$\begin{aligned} \ln(H(n)) &\approx \int_1^n x \ln(x)\, \mathrm{d}x + \frac{n \ln(n)}{2} + \frac{B_2}{2!} \ln(n) \\ &= \frac{n^2 \ln(n)}{2} + \frac{n \ln(n)}{2} - \frac{n^2}{4} + \frac{1}{4} + \frac{1}{12} \ln(n) + c \end{aligned} \tag{10}$$

where $B_2 = \frac{1}{6}$ is the second Bernoulli number, and c is a constant that represents the remainder error when expressing the Equation 9 sum as an integral in Equation 10. This remainder operation is standard in Euler Maclaurin summation theorem theory, and will be discussed in detail in the next post. For now, it is also important to note that for $f(x) = x \ln(x)$ that the second and higher derivatives fall off increasingly quickly for large n, thus vanishing in the asymptotic expression. We can re-exponentiate Equation 10, and attain the asymptotic expression:

$$H(n) = A n^{\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}} e^{-\frac{n^2}{4}} \tag{11}$$

where $A = 1.28243$ is the Glaisher-Kinkelin Constant, and is the best fit for the remainder expression ($A = e^{\frac{4c+1}{4}}$) in Equation 10. Now, we give the asymptotic expression from Equation 4:

$$\lim_{n \to \infty} e^{n^{1-\frac{1}{n-1}}} > \frac{\left(n^{1-\frac{1}{n-1}}\right)^{\frac{n-1}{2}}}{\sqrt{2\pi}} n^{-\frac{n}{2}+1} e^{\frac{3}{4}n} = \frac{e^{\frac{3}{4}n}}{\sqrt{2\pi}} \tag{12}$$

Next post will be about proving the Euler Maclaurin summation theorem. Also to be discussed is the geometric mean of the harmonic series, which is also interesting as well as Equation 12. This concludes this blog post.