Title: Inverse Function Sequences and Series
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In this post, I will show a method for computing the inverse function given the original function's power series coefficients. Since having discovered the patterning yielding correct calculation, I have found a Mathworld page that highlights the rigourous equation used to complete this task. I will take a less rigourous route, but one that shows clearly the origin of the pattern highlighting the computation of the nth series term of the inverse function. To begin, first we consider a domain over on which an inverse is well-defined. Then we define the two series', where without loss of generality, we set the lowest order term equal to 0:

$$\begin{aligned} f\left(x\right) &= \sum_{n=1}^{\infty} a_n x^n \\ f^{-1}\left(x\right) &= \sum_{n=1}^{\infty} c_n x^n \end{aligned} \tag{1}$$

The defining equation of the inverse function is the following:

$$f^{-1}\left(f\left(x\right)\right) = x \tag{2}$$

Thus, we consider the following problem:

$$c_n \left(\sum_{k=1} a_k x^k\right)^n = \delta_{n1} x^n \tag{3}$$

This problem definition is as follows: the nth power term is composed completely by lower order terms. For instance, for $n = 1, 2$:

$$\begin{aligned} c_1 a_1 x = x, \implies c_1 = \frac{1}{a_1} \\ c_2 a_1^2 x^2 + c_1 a_2 x^2 = 0x^2, \implies c_2 = -c_1 \frac{a_2}{a_1^2} = -\frac{a_2}{a_1^3} \end{aligned} \tag{4}$$

Through counting the partitioned combinations of terms for $n = 3, 4, 5, 6, 7$, I attained the following results:

$$
\begin{aligned}
c_3 &= -\frac{a_3}{a_1^4} + 2\frac{a_2^2}{a_1^5} \\
c_4 &= -\frac{a_4}{a_1^5} + 5\frac{a_2 a_3}{a_1^6} - 5\frac{a_2^3}{a_1^7} \\
c_5 &= -\frac{a_5}{a_1^6} + 6\frac{a_2 a_4}{a_1^7} + 3\frac{a_3^2}{a_1^7} - 21\frac{a_2^2 a_3}{a_1^8} + 14\frac{a_2^4}{a_1^9} \\
c_6 &= -\frac{a_6}{a_1^7} + 7\frac{a_2 a_5}{a_1^8} + 7\frac{a_3 a_4}{a_1^8} - 28\frac{a_2^2 a_4}{a_1^9} - 28\frac{a_2 a_3^2}{a_1^9} + 84\frac{a_2^3 a_3}{a_1^{10}} - 42\frac{a_2^5}{a_1^{11}} \\
c_7 &= -\frac{a_7}{a_1^8} + 8\frac{a_2 a_6}{a_1^9} + 8\frac{a_3 a_5}{a_1^9} + 4\frac{a_4^2}{a_1^9} - 36\frac{a_2^2 a_5}{a_1^{10}} \\
&\quad - 72\frac{a_2 a_3 a_4}{a_1^{10}} - 12\frac{a_3^3}{a_1^{10}} + 120\frac{a_2^3 a_4}{a_1^1 1} + 180\frac{a_2^2 a_3^2}{a_1^{11}} - 330\frac{a_2^4 a_3}{a_1^1 2} + 132\frac{a_2^6}{a_1^{13}}
\end{aligned} \tag{5}
$$

As the term numbers grow, a pattern emerges that is relatively straightforward to spot. First, the number of powers of a_1 in the denominator is always one greater than the sum of the powers in the numerator. As the powers of the denominator grow from a_1^{n+1} to a_1^{2n-1}, we find that the powers in the numerator increase by one each time, such that the powers of the denominator always remain one greater. Each time the numerator / denominator power increases, the sign of the term alternates. The next pattern becomes more evident in c_n as n increases. The coefficient of each term can be written as follows:

$$
c_n = \frac{(-1)^n}{n!} \sum_{m=n}^{2n-2} \left(-\frac{1}{a_1}\right)^{m+1} m! \sum_{q_m=1}^{N_{mn}} \prod_{k=2}^{n} \left(\frac{a_k^{p_{q_m k}}}{(p_{q_m k})!}\right) \tag{6}
$$

Here, N_{mn} corresponds to the number of term combinations that give the mth power with $m-n+1$ terms. Here, $p_{q_m k}$ corresponds to the power of a_k in the term in question (for many k for a given m and q_m, $p_{q_m k} = 0$). In general, the following patterns hold true:

$$
\begin{aligned}
m &= \sum_{k=2}^{n} k p_{q_m k} \\
n &= 1 + \sum_{k=2}^{n} (k-1)\, p_{q_m k} \\
N_{mn} &= \prod_{t=1}^{m-n} \sum_{i_t = i_{t-1}}^{\lfloor \frac{m-\sum_{s=1}^{t-1} i_s}{m-n-t+2} \rfloor} (1)
\end{aligned} \tag{7}
$$

where $i_0 = 2$, $\sum_{s=1}^{0} f(s) = 0$, and $\prod_{t=1}^{0} f(t) = 1$. All that is left is to determine $p_{q_m k}$, which is related to the integer partition function (i.e. the combinations of a_k terms that give power n when substituting $f(x)$ into the inverse series - Equation 3). While the formal expression relating the partition function to the number and nature of combinations is difficult to define, an algorithm that reflects the thinking used to derive N_{mn} can be used to find all combinations needed to derive the $p_{q_m k}$ powers needed to compute the coefficients for each q_m term associated with each power m numerator. First, take the $m-n+1$ terms needed to make the given mth power numerator term. Make the first term a_2. Then, if the next term is not the last, make it a_2 as well, until you arrive at the last term, which you make $a_{m-2(m-n)} = a_{2n-m}$ - the difference between the total power m and the sum of the powers of the previous terms. You then increment the second last term by 1, and subtract 1 from the last term to make another term *if* the two previous k terms are not within 1 of each other. One then does this continuously until the absolute difference between the last factor k and the second last factor a_k is at most 1. Then go back to the third last term and repeat the process, where the terms are ascending in a_k from left to right. This algorithm can be seen on display when moving from right to left for each of the computed terms for each numerator power m in Equation 4. Then, one can construct an algorithm with this logic to compute the inverse function for any power series $f(x)$ for which an inverse is defined.

First, it is worth noting that this method can be generalized to $a_0 \neq 0 \neq c_0$, by translating the inverse horizontally by the amount that the original function is translated vertically (that is, if $g(x) = g(0) + f(x)$, then $g^{-1}(x) = f^{-1}(x - g(0))$). Also, there is a problem of if $a_1 = 0$, such as in $f(x) = 1 - \cos(x)$. In this case, this means that $f(0) = 0$, and $f'(0) = 0$, such that there is an extrema at $x = 0$. This means that $\lim_{x \to 0} \left(f^{-1}(x)\right)' = \infty$ (though noteably, this is sloppy use of limit notation). Nonetheless, the idea is simple: $\frac{\partial f^{-1}(f(x))}{\partial f(x)} = \frac{1}{f'(x)}$. So in this example, $f(x)$ has $f^{-1}(x)$ that is a diverging power series (or more simply, can be written as a non-integer power series) - in fact, every derivative of the inverse function diverges at $x = 0$, except the inverse itself. There is a simple reason for this - two algorithmic points become important:

- One can use the differential method to derive the inverse function - let $y = 1 - \cos(x)$. Then:

$$\frac{\partial f^{-1}(y)}{\partial y} = \frac{1}{\sin(x)} = \frac{1}{\sqrt{y(2-y)}} \implies f^{-1}(y) = \arccos(1-y) \tag{8}$$

  as expected.

- The $f^{-1}(x) = \arccos(1-x)$ solution represents a power series that cannot be written in a converging way purely in terms of powers of x. To see why this is the case:

$$\begin{aligned}
\arccos(1-x) &= \frac{\pi}{2} - \sum_{n=0}^{\infty} \left(\frac{(2n)!}{2^{2n}(n!)^2} \frac{(1-x)^{2n+1}}{2n+1} \right) \\
&= \frac{\pi}{2} - \sum_{n=0}^{\infty} \left(\frac{(2n)!}{2^{2n}(n!)^2} \frac{1}{2n+1} \sum_{k=0}^{2n+1} \binom{2n+1}{k} (-x)^k \right) \\
&= \sum_{n=0}^{\infty} \left(\frac{(2n)!}{2^{2n}(n!)^2} x \right) - \sum_{n=0}^{\infty} \sum_{k=2}^{2n+1} \left(\frac{(2n)!\,(2n)_{k-1}}{2^{2n}(n!)^2} \frac{(-x)^k}{k!} \right)
\end{aligned} \tag{9}$$

  The coefficient of every power of x in Equation 9 diverges, but the series as a whole converges for all values of $|1-x| < 1$. However, writing down the inverse series as a series inversion is not feasible, as the series inversion tells one nothing about *how* the coefficient of each term diverges (which determines precisely how the terms will sum into each other (and cancel) to produce a finite value). Thus, the differential method (rather than the series inversion method) must be used to find the inverse when $a_0 = a_1 = 0$ for the original function.

One idea remains that I would like to bring attention to: what is the long-term behaviour of the sequence resulting from having a function consisting of a single polynomial power, inverting it, taking the derivative, taking absolute value, then inverting again, then taking the derivative again, then taking absolute value, and so on. For example, consider $f(x) = \frac{1}{x}$ (the absolute value notation will be omitted, but *remember* it is there). Then, $f^{-1}(x) = \frac{1}{x}$. Then, $\left(f^{-1}(x)\right)' = \frac{1}{x^2}$, and $\left(\left(f^{-1}(x)\right)'\right)^{-1} = x^{-\frac{1}{2}}$ and $\left(\left(\left(f^{-1}(x)\right)'\right)^{-1}\right)' = \frac{1}{2}x^{-\frac{3}{2}}$ and $\left(\left(\left(\left(f^{-1}(x)\right)'\right)^{-1}\right)'\right)^{-1} = \left(\frac{1}{2}\right)^{\frac{2}{3}} x^{-\frac{2}{3}}$ etc. The rth term is:

$$\omega_r = \frac{\prod_{\ell=3}^{r-1} F_\ell^{\frac{F_{\ell-1}}{F_{r+1}}}}{(F_r x)^{\frac{F_r}{F_{r+1}}}}$$
$$\lim_{r\to\infty} (\omega_r)_{x\to 1} = \lim_{r\to\infty} \frac{\prod_{\ell=3}^{r-1} F_\ell^{\frac{\sqrt{5}F_{\ell-1}}{\phi^{r+1}}}}{\left(\frac{\phi^r}{\sqrt{5}}\right)^{\frac{1}{\phi}}} \approx 0.45904038 \tag{10}$$

where ω_r is the rth inverse function in the sequence, F_r is the rth Fibonacci Number ($F_r = 1, 1, 2, 3, 5, 8...$), where where for large r, $F_r \approx \frac{\phi^r}{\sqrt{5}}$, where $\phi = \frac{1+\sqrt{5}}{2} \approx 1.61$ is the golden ratio. The second line constant was computed by MATLAB. This constant emerges from the sequence assuming a starting power of $a = -1$, for $f(x) = x^a$. More sequence starting points will be investigated in future posts. Critically, the power of x approaching the golden ratio as r becomes large is intuitive because $\phi = \frac{1}{\phi-1}$, such that taking the derivative and then the inverse makes no change to the power of x. The Fibonacci Numbers arise since taking the derivative and inverting gives a power for the next derivative of $p_r = -\frac{1}{\frac{F_{r-1}}{F_r}+1} = -\frac{F_r}{F_{r+1}}$, thus the power evolution sequence propagates within the Fibonacci Numbers as $F_{r+1} = F_r + F_{r-1}$.

This idea represents something that I have generalized to a function of a variable starting power of the initial function. Define $\omega_0 = x^{-a}$, such that $\omega_1 = a^{\frac{1}{a+1}} x^{-\frac{1}{a+1}}$, $\omega_2 = a^{\frac{1}{a+2}} \left(\frac{1}{a+1}\right)^{\frac{a+1}{a+2}} x^{-\frac{a+1}{a+2}}$, and $\omega_3 = a^{\frac{1}{2a+3}} \left(\frac{1}{a+1}\right)^{\frac{a+1}{2a+3}} \left(\frac{a+1}{a+2}\right)^{\frac{a+2}{2a+3}} x^{-\frac{a+2}{2a+3}} = a^{\frac{1}{2a+3}} (a+1)^{\frac{1}{2a+3}} ((a+2)x)^{-\frac{a+2}{2a+3}}$. This pattern emerges to:

$$\omega_r = \frac{a^{\frac{1}{F_r a + F_{r+1}}} \prod_{\ell=3}^{r} \left((F_{\ell-2}a + F_{\ell-1})^{\frac{F_{\ell-3}a + F_{\ell-2}}{F_r a + F_{r+1}}} \right)}{((F_{r-1}a + F_r)x)^{\frac{F_{r-1}a + F_r}{F_r a + F_{r+1}}}}, r > 0 \tag{11}$$

This equation is directly corresponding to Equation 10 but with $r \to r+1$ - put differently, Equation 10 applies for $r = 0$, while Equation 11 applies only for $r \geq 1$ (assuming that $\omega_0 \geq x^{-a}$ and $F_0 = 0$). Numerical programming suggests that the constant (which is independent of the value of $a > 0$, at least for $\mid a \mid < 10$) for the limit is:

$$\lim_{r\to\infty} \omega_r \mid_{x\to 1} = 0.4590403846822 \tag{12}$$

To be analytical, assume that $r \to \infty$ and $a \sim \frac{1}{r}$ - then, take $F_r \to \frac{\phi^r}{\sqrt{5}}$:

$$\lim_{r\to\infty} \left(\omega_r\left(a \to \frac{1}{r}\right)\right)_{x\to 1} = \lim_{r\to\infty} \frac{\left(\frac{1}{r}\right)^{\frac{\sqrt{5}r}{\phi^r + r\phi^{r+1}}} \prod_{\ell=3}^{r} \left(\left(\frac{F_{\ell-2}}{r} + F_{\ell-1}\right)^{\frac{F_{\ell-3} + rF_{\ell-2}}{\phi^r + r\phi^{r+1}}\sqrt{5}} \right)}{\left(\left(\frac{\phi^{r-1}}{\sqrt{5}r} + \frac{\phi^r}{\sqrt{5}}\right)\right)^{\frac{\phi^{r-1} + r\phi^r}{\phi^r + r\phi^{r+1}}}} \tag{13}$$

Manipulating the product:

$$\lim_{r\to\infty}\left(\omega_r\left(a\to\frac{1}{r}\right)\right)_{x\to1}=\lim_{r\to\infty}\frac{\left(\frac{1}{r}\right)^{\frac{\sqrt{5}}{\phi^{r+1}}}\prod_{\ell=3}^{r}\left(\left(\frac{F_{\ell-2}}{r}+F_{\ell-1}\right)^{\frac{\sqrt{5}F_{\ell-2}}{\phi^{r+1}}}\right)}{\left(\frac{\phi^{r-1}}{\sqrt{5}r}+\frac{\phi^r}{\sqrt{5}}\right)^{\frac{1}{\phi}}} \tag{14}$$

Evaluating:

$$\lim_{z\to\infty}\left(\frac{1}{z}\right)^{\frac{\sqrt{5}}{\phi^{z+1}}}=e^{-\frac{\sqrt{5}}{\phi}\lim_{z\to\infty}\left(\frac{\ln(z)}{\phi^z}\right)}=e^{-\frac{\sqrt{5}}{\phi}\lim_{z\to\infty}\left(\frac{1}{z\ln(\phi)\phi^z}\right)}=1 \tag{15}$$

Carrying out this operation and some other minor revisions, we get:

$$\lim_{r\to\infty}\left(\omega_r\left(a\to\frac{1}{r}\right)\right)_{x\to1}=\lim_{r\to\infty}\frac{\left(\prod_{\ell=3}^{r}\left(\left(\frac{F_{\ell-2}}{r}+F_{\ell-1}\right)^{F_{\ell-2}}\right)\right)^{\frac{\sqrt{5}}{\phi^{r+1}}}}{\left(\frac{\phi^{r-1}}{\sqrt{5}r}+\frac{\phi^r}{\sqrt{5}}\right)^{\frac{1}{\phi}}} \tag{16}$$

Now, if we presume that the only terms that matter in the product are those for which $F_\ell\approx\frac{\phi^\ell}{\sqrt{5}}$, then we get:

$$\lim_{r\to\infty}\left(\omega_r\left(a\to\frac{1}{r}\right)\right)_{x\to1}=\lim_{r\to\infty}\frac{\prod_{\ell=3}^{r}\left(\left(\frac{\phi^{\ell-2}}{r\sqrt{5}}+\frac{\phi^{\ell-1}}{\sqrt{5}}\right)^{\phi^{\ell-r-3}}\right)}{\left(\frac{\phi^{r-1}}{r\sqrt{5}}+\frac{\phi^r}{\sqrt{5}}\right)^{\frac{1}{\phi}}} \tag{17}$$

Rearranging:

$$\begin{aligned}
\lim_{r\to\infty}\left(\omega_r\left(\frac{1}{r}\right)\right)_{x\to1}&=\lim_{r\to\infty}\frac{\prod_{\ell=3}^{r}\left(\left(\frac{\phi^{\ell-1}}{\sqrt{5}}\right)^{\phi^{\ell-r-3}}\left(1+\frac{1}{r\phi}\right)^{\phi^{\ell-r-3}}\right)}{\left(\frac{\phi^r}{\sqrt{5}}\right)^{\frac{1}{\phi}}\left(1+\frac{1}{r\phi}\right)^{\frac{1}{\phi}}}\\
&=\lim_{r\to\infty}\prod_{\ell=3}^{r}\left(\left(\frac{\phi^{\ell-1}}{\sqrt{5}}\right)^{\phi^{\ell-r-3}}\left(1+\frac{\phi^{\ell-r-4}}{r}\right)\right)\left(\frac{\phi^r}{\sqrt{5}}\right)^{-\frac{1}{\phi}}\left(1-\frac{1}{r\phi^2}\right)\\
&=\lim_{r\to\infty}\prod_{\ell=3}^{r}\left(\phi^{(\ell-1)\phi^{\ell-r-3}}\left(\frac{1}{\sqrt{5}}\right)^{\phi^{\ell-r-3}}\right)\left(\frac{\phi^r}{\sqrt{5}}\right)^{-\frac{1}{\phi}}\left(1-\frac{1}{r\phi^2}\right)\\
&=\lim_{r\to\infty}\left(\phi^{\sum_{\ell=3}^{r}(\ell-1)\phi^{\ell-r-3}}\left(\frac{1}{\sqrt{5}}\right)^{\sum_{\ell=3}^{r}\phi^{\ell-r-3}}\left(\frac{\phi^r}{\sqrt{5}}\right)^{-\frac{1}{\phi}}\right)\\
&=\lim_{r\to\infty}\left(\phi^{\frac{1}{\phi^{r+2}}\left(-\left(\phi^2-2\phi\right)\mathrm{Li}_{-1}(\phi)+\frac{\phi^r(r\phi-r-\phi)}{(\phi-1)^2}-\frac{r}{\phi}\right)}\left(\frac{1}{\sqrt{5}}\right)^{-\frac{1}{\phi^{r+2}}\frac{\phi^2-\phi^r}{\phi-1}-\frac{1}{\phi}}\right)\\
&=\lim_{r\to\infty}\left(\phi^{r\left(\frac{\phi-1-\frac{\phi}{r}-\phi(1-\phi)^2}{\phi^2(1-\phi)^2}\right)}\left(\sqrt{5}\right)^{-\frac{1}{\phi^2(\phi-1)}+\frac{1}{\phi}}\right)\\
&=\left(\frac{1}{\phi}\right)^{\frac{1}{\phi(1-\phi)^2}}=(\phi-1)^{\phi}=0.4590403846822343
\end{aligned} \tag{18}$$

which is corresponds nicely to the numerical result provided in Equation 12, presumably within numerical error. However, there is one concept to still be worked out - it is the numerical observation that Equation 11 seems to be independent of a. This observation allows for the result in Equation 18 to work out as an expression of the product value for *all* values of a. The idea for how to prove this alludes me currently. This concludes this blog post.