Title: Analysis of Sums Arising from Riemann Zeta Function

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While examining integrals of the Riemann Zeta function along the real number line (such as the Fourier Transform), I came across some very useful methods to test for the convergence of a series containing logarithms as part of its coefficients. For example, for $s = \sigma + it$ for $t = 0$ and $\sigma > 1$, we have:

$$\begin{aligned} \int_1^\infty \left(\zeta\left(\sigma\right) - 1\right) \mathrm{d}\sigma &= \sum_{n=2}^\infty \int_1^\infty \mathrm{d}\sigma \frac{1}{n^\sigma} \\ &= \sum_{n=2}^\infty \int_1^\infty \mathrm{d}\sigma e^{-\sigma \ln(n)} \\ &= \sum_{n=2}^\infty \frac{1}{n \ln\left(n\right)} \end{aligned} \tag{1}$$

I had for long held the (ignorant) view that any sum that proceeds more slowly than the harmonic series ($\sum_{n=1}^\infty \frac{1}{n}$) must converge as an infinite series. Certainly, anything that increases faster than the harmonic series will diverge. But, it turns out that it is well known (at least in general mathematical communities, rather than the engineering physics communities that I have come from) that there is *no slowest converging or slowest diverging sum*, and indeed there is no defined boundary between convergence and divergence. This is well exemplified in the late Walter Rudin's book "Principles of Mathematical Analysis". This idea is well demonstrated through the following example I found on the worldwide web. All of the following sums diverge, increasingly more slowly:

$$\begin{aligned} d_1^k &= \sum_{n=k}^\infty \frac{1}{n} \\ d_2^k &= \sum_{n=k}^\infty \frac{1}{n \ln\left(n\right)} \\ d_3^k &= \sum_{n=k}^\infty \frac{1}{n \ln\left(n\right) \ln\left(\ln\left(n\right)\right)} \\ d_4^k &= \sum_{n=k}^\infty \frac{1}{n \ln\left(n\right) \ln\left(\ln\left(n\right)\right) \ln\left(\ln\left(\ln\left(n\right)\right)\right)} \end{aligned} \tag{2}$$

Assuming that k is chosen such that d_n is well defined in the sense of the logarithm (no negative arguments - all terms remain real). We will see soon enough why the functions in this example diverge - however, it is clear that if we propose one slowly diverging series, we can always multiply each term by an additional logarithm term that will make it diverge even more slowly. On the contrary, there is also an easy example showing that we can write arbitrarily slower converging sums by the following:

$$\begin{aligned} c_1^k &= \sum_{n=k}^{\infty} \frac{1}{n^2} \\ c_2^k &= \sum_{n=k}^{\infty} \frac{1}{n \ln^2(n)} \\ c_3^k &= \sum_{n=k}^{\infty} \frac{1}{n \ln(n) \ln^2(\ln(n))} \\ c_4^k &= \sum_{n=k}^{\infty} \frac{1}{n \ln(n) \ln(\ln(n)) \ln^2(\ln(\ln(n)))} \end{aligned} \tag{3}$$

To see why these sums are increasingly slowly diverging or converging, we define Cauchy's Condensation Test, which is described on Mathworld at https://mathworld.wolfram.com/CauchyCondensationTest.html. In this test, if there exists a sequence such that $a_{n+1} \leq a_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ converges *if and only if* $\sum_{k=1}^{\infty} 2^k a_{2^k}$ converges. This means that the convergence behaviour of the original series and Cauchy's series are equivalent. Cauchy's series is useful in the case that logarithms occur in a_n, since the inputs to the series terms are exponents of 2. In this case, the second line of Equation 2 clearly diverges, since:

$$\sum_{k=1}^{\infty} 2^k a_{2^k} = \sum_{k=0}^{\infty} \frac{2^k}{2^k \ln(2^k)} = \sum_{k=1}^{\infty} \frac{1}{k \ln(2)} \tag{4}$$

which diverges like the harmonic series. Likewise, the other sums in Equation 2 diverge. Using the same test, the second line of Equation 3 can be shown to converge, since:

$$\sum_{k=1}^{\infty} 2^k a_{2^k} = \sum_{k=0}^{\infty} \frac{2^k}{2^k \ln^2(2^k)} = \sum_{k=1}^{\infty} \frac{1}{(k \ln(2))^2} \tag{5}$$

which converges in the same way that the Riemann Zeta function converges with argument $\sigma = 2$. This test can be used to show that all sums in Equation 3 converge, albeit increasingly slowly.

Now that we have shown that there is no well-defined "boundary" between convergent and divergent sums, we will demonstrate an interesting result of integrating Dirichlet Eta function over the real numbers between $1 < \sigma < \infty$. Based on a similar computational method used for Equation 1, we can write (where the last line is valid for $k \geq 0$):

$$\begin{aligned} b_k &= \int_1^{\infty} \mathrm{d}\sigma_k \prod_{i=2}^{k} \int_{\sigma_i}^{\infty} \mathrm{d}\sigma_{i-1} \left(\eta(\sigma_1) - 1\right) \\ &= \sum_{n=2}^{\infty} \int_1^{\infty} \mathrm{d}\sigma_k \prod_{i=2}^{k} \int_{\sigma_i}^{\infty} \mathrm{d}\sigma_{i-1} \left(\frac{(-1)^{n-1}}{n^{\sigma_1}} \right) \\ &= \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n \ln^k(n)} \end{aligned} \tag{6}$$

Thus, $-\sum_{k=0}^{\infty} \frac{b_k}{k!} = \sum_{n=2}^{\infty} \left((-1)^n \frac{e^{\frac{1}{\ln(n)}}}{n} \right) = 1.59095...$ according to Mathematica. Interestingly, this result is within 5% of $2\eta(2)$. This concludes this blog post.