Title: Circular Intersection Geometry
Author: Josh Myers

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In this post, we will explore the properties of the maximum area triangle that can circumscribed inside a circle of fixed radius R. The problem visual is given in Figure 1. The question posed is

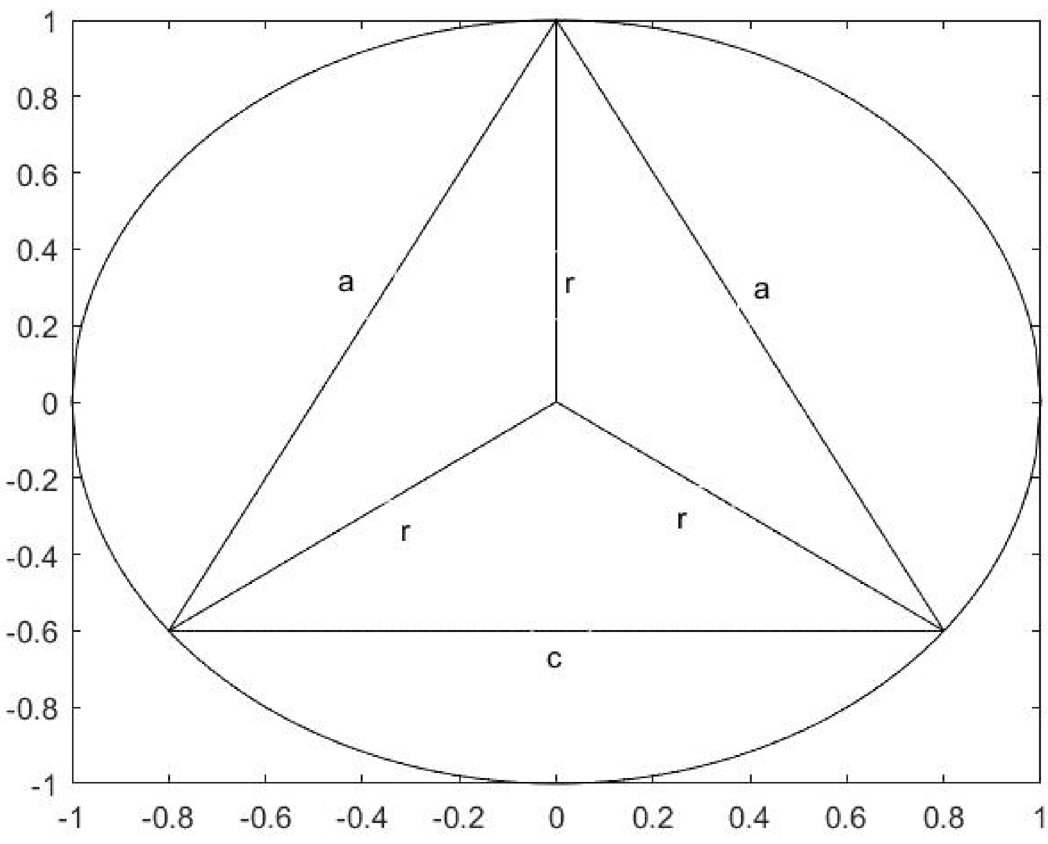

Figure 1: The geometry to be analyzed. In this image, the circle is the unit circle, and the triangle is isosceles.

the following: what is the properties of the circumscribed triangle that occupies the largest area A within the circle keeping circle radius r and triangle perimeter P fixed? Which perimeter P gives the largest area A of the previous solutions, keeping r fixed? Finally, how does the ideal perimeter P change with r? Lastly, of the solutions A for some P, which gives the maximum $\frac{A}{Pr}$ ratio?

To simplify this problem, we will invoke some symmetry. Consider first Heron's formula where a, b, and c are the three side lengths of the triangle:

$$A = \sqrt{\frac{P}{2}\left(\frac{P}{2} - a\right)\left(\frac{P}{2} - b\right)\left(\frac{P}{2} - c\right)} \tag{1}$$

To understand why the maximum area circumscribed triangle must be isosceles, we consider the case where $b = a - \Delta$. Generally, $c = P - a - b$. To keep the perimeter fixed, we have that $a \to a + \Delta$. Then:

$$A = \sqrt{\frac{P}{2}\left(\frac{P}{2} - a - \Delta\right)\left(\frac{P}{2} - a + \Delta\right)\left(2a - \frac{P}{2}\right)} \tag{2}$$

Now the task is simple - what value of Δ optimizes A? We take the derivative of A with respect to Δ, set it equal to 0, then solve for Δ. We find only a single solution:

$$\begin{aligned} \left(2a - \frac{P}{2}\right)\left(-a - \Delta + \frac{P}{2}\right)P - \left(2a - \frac{P}{2}\right)\left(-a + \Delta + \frac{P}{2}\right)P = 0 \\ \implies \Delta = 0 \end{aligned} \tag{3}$$

This solution is a global maximum, as can be seen by taking the 2nd derivative and substituting $\Delta = 0$:

$$\left(\frac{\mathrm{d}^2 A}{\mathrm{d}\Delta^2}\right)_{\Delta \to 0} = -\frac{\left(2a - \frac{P}{2}\right)P}{\sqrt{2\left(2a - \frac{P}{2}\right)\left(-a + \frac{P}{2}\right)^2 P}} \leq 0 \tag{4}$$

where the global maximum is an implication of the triangle inequality, since $2a \geq \frac{P}{2}$ for *all* isosceles triangles. Having carried out verification analysis to determine that $\Delta = 0$ gives the (isosceles) triangle of maximum area A given a fixed perimeter P, we can now express the simplified maximum area in terms of the elementary geometric formula $A_{\text{simp}} = \frac{1}{2}b_o h$, where b_o is the base and h the height. As given in Figure 1, it is clear that $b_o = c$ and $h = r + \sqrt{r^2 - \left(\frac{c}{2}\right)^2}$. Thus, in this analysis, we leave P out of the calculation (it leads to confounding roots, as negative side lengths can quickly come into the calculation undetected!). Now take the derivative of the area with respect to c, set it to 0, and solve for c, the base of the isosceles triangle:

$$\begin{aligned} A_{\text{simp}} &= \frac{1}{2}c\left(r + \sqrt{r^2 - \left(\frac{c}{2}\right)^2}\right) \\ \implies \frac{\mathrm{d}A_{\text{simp}}}{\mathrm{d}c} &= \frac{1}{2}\left(r + \sqrt{r^2 - \frac{c^2}{4}}\right) - \frac{c^2}{8\sqrt{r^2 - \frac{c^2}{4}}} = 0 \\ \implies c &= 0, c = \pm\sqrt{3}r \end{aligned} \tag{5}$$

where the solution we search for is found in $c = \sqrt{3}r$. This solution gives the following implications:

$$\begin{aligned} a &= \sqrt{3}r \\ P &= 3\sqrt{3}r \\ A &= \frac{3\sqrt{3}}{4}r^2 \end{aligned} \tag{6}$$

the equilateral triangle. Now we can use the same strategy to calculate the maximum $\frac{A_{\text{simp}}}{rP}$, where $P = 2a + c$, and $a = \sqrt{\left(\frac{c}{2}\right)^2 + \left(r + \sqrt{r^2 - \left(\frac{c}{2}\right)^2}\right)^2}$:

$$\frac{\mathrm{d}\left(\frac{A_{\text{simp}}}{Pr}\right)}{\mathrm{d}c} = \frac{\sqrt{r\left(2r + \sqrt{4r^2 - c^2}\right)} - c}{4r\sqrt{4r^2 - c^2}} = 0 \implies c = \pm\sqrt{3}r \tag{7}$$

where we again find the same equilateral triangle solution for optimizing the ratio of area to triangle perimeter.

Many questions remain unanswered - for example, if we fix P and r, then how does the optimal area vary with these parameters (to be clear, only the global maximum area has been found, where a P does not vary - is fully determined). As well, how does this method apply to solving for the optimal area of shapes with more than 3 vertices, such as cyclic quadrilaterals. I have great interest in carrying out a similar calculation for cyclic quadrilaterals. This concludes this blog post.