Title: Variations on the Euler-Maclaurin Formula
Author: Josh Myers

July 1, 2025

This post follows up on the post I made recently about a variation of the Euler-Maclaurin formula. That post involved a recursive function definition which is as follows:

$$F_k''(x) = -\frac{1}{2}\sum_{w=0}^{2}(-2)_{2-w}\left(B_w(1)\,F_{k-1}^{(w+1)}(x_k+1) - B_w(0)\,F_{k-1}^{(w+1)}(x_k)\right) \tag{1}$$

where $k \in \mathbb{W}$ and $F_0''(x) = -\frac{1}{2}f''(x)$, where $f''(x)$ is the second derivative of the summation function in the Euler-Maclaurin formalism (see the March 15th 2025 post). To initially study the progression of this recursive sequence, I substitute $f''(x) = e^x$. The following developments ensue:

$$\begin{aligned} F_0''(x) &= -\frac{1}{2}e^x \\ F_k''(x) &= -\frac{1}{2}\left(\frac{19-7e}{12}\right)^k e^x \end{aligned} \tag{2}$$

where $\frac{7e-19}{12} \approx 0.00233$. As can be seen, F_k scales down by a factor of about 500 for every k. From the previous post, it is well known that:

$$\mid R_2 \mid < \frac{2\zeta(2)}{(2\pi)^2}\int_1^N \mathrm{d}x_0 \mid f^{(2)}(x_0) \mid \tag{3}$$

Thus, based on the definition that $F_0''(x) = -\frac{1}{2}f''(x)$, we can reformulate this inequality for F_k'':

$$\begin{aligned} \mid R_2^{(k)} \mid &< \frac{1}{6}\int_1^{N-k}\mathrm{d}x_k \mid F_k''(x_k) \mid \\ &< \frac{1}{6}\left(\frac{7e-19}{12}\right)^k\left(e^{N-k} - e\right) \end{aligned} \tag{4}$$

Note that this expression is only so easy because of e^x is everywhere greater than 0, so doing the integral of the absolute value is not a special problem. Solving for F_k'' is substantially more challenging for functions that are not e^x, since the derivatives bring in new operations that cannot be so compactly represented in terms of a general k - for instance, the power term $F_0''(x) = -\frac{1}{2}x^n$ is overwhelming. However, there are other function "starting points" that produce compact results - to be discussed next. Overall, perhaps it can be gleaned from this example that the Bernoulli numbers play a mysterious role in scaling down these recursive remainder terms that come up as governed by Equation 1. Based on the formula computed in the last post, it is clear that finite sums can be evaluated in terms of this method - however, there are challenges that arise. For instance, Equation 1 solves for only F_k'', and calculation must be done to attain F_k''' and F_k' before the next F_{k+1}'' can be evaluated. This means that every F_k'' must have an anti-derivative in order to carry out the next order of calculation - an obvious constraint.

Let's solve for $F_k''(x)$ for another everywhere positive function - consider $F_0'' = -\frac{1}{2}\sin^2(x)$. In this case, we get the following:

$$F_k''(x) = \frac{1}{4}\left(\frac{3\cos(1) - 2\sin(1)}{3}\right)^k \cos(2x + k) \tag{5}$$

where $\frac{3\cos(1)-2\sin(1)}{3} = -0.0206784$, so the integrated function scales down by a factor of about 50 each iteration of the Euler-Maclaurin formula variation. These scaling factors are very interesting to examine, as they give clear indication of the "value" provided by carrying out another Euler-Maclaurin summation iteration. The interesting fact is that for both Equation 2 and Equation 5, the general summation function remains within the same family for all integer $k > 0$. Categorically, Equation 1 (which governs the evolution of $F_k''(x)$) is a differential difference equation - a special case of the functional differential equation. The simple results that we achieve with Equation 2 and Equation 5 are not to be found for any $F_0''(x)$ - for example, consider $F_0''(x) = x^n$. However, another interesting pattern is gleaned by numerical study of $F_0''(x) = x^n$ - that is, that every kth iteration leads to a detailed and not simple formula that goes to 0 if $n < 4k$. This makes the Euler-Maclaurin expansion studied in the last post highly efficient for computing the power sum with 0 remainder. To prove this identity, consider first that from Equation 1, we get the following for the highest order polynomial terms for $F_{k-1}''(x) = x^n$ such that $F_k''(x) = a_1 x^n + a_2 x^{n-1} + a_3 x^{n-2} + a_4 x^{n-3} \ldots$:

$$\begin{aligned}
a_1 &= -\left(\frac{1}{4}\right)2 + \frac{1}{2}\frac{1}{n+1}\binom{n+1}{1} = 0 \\
a_2 &= -\left(\frac{1}{4}\right)\binom{n}{1} + \frac{1}{2}\frac{1}{n+1}\binom{n+1}{2} = 0 \\
a_3 &= \frac{n}{24}\binom{n-1}{1} - \frac{1}{4}\binom{n}{2} + \frac{1}{2}\frac{1}{n+1}\binom{n+1}{3} = \frac{(n)_2}{24} - \frac{(n)_2}{8} + \frac{(n)_2}{12} = 0 \\
a_4 &= \frac{n}{24}\binom{n-1}{2} - \frac{1}{4}\binom{n}{3} + \frac{1}{2}\frac{1}{n+1}\binom{n+1}{4} = \frac{(n)_3}{48} - \frac{(n)_3}{24} + \frac{(n)_3}{48} = 0
\end{aligned} \tag{6}$$

where $(n)_p$ is the falling factorial of order p. This very general identity means that for every nth power term substituted into Equation 1, the resulting surviving polynomial terms will be $(n-4)$th power and less. Then, if these terms are substituted again for the next $k+1$, then the highest surviving term will be $(n-8)$th power and less. This trend continues, with computationally sophisticated general expressions for the a_m surviving coefficients, but with certainty that each time F_k is computed, the power of the surviving terms will be decreased by 4. This is simple for finite polynomials - however, for functions represented by infinite power series, F_k can become computationally complicated to determine in terms of k. However, Equation 2 and Equation 5 are examples of simplified functions in which F_k'' can be computed generally in terms of a very similar family of functions expressible as infinite power series'. It will be interesting to investigate these identities further to come to an understanding of a general trend in starting functions for which F_k'' can be reliably expressed in terms of k. I'm also interested in trying this decomposition in terms of Riemann Zeta like series - though the analytic continuation of this function for $0 < \mathbb{R}(s) < 1$ is often considered intractable, I am curious what results will appear with negative integer polynomial powers (for $\mathbb{R}(s) > 1$). This will be the topic of my next post. This concludes this blog post.