Title: Riemann Hypothesis: Discussion of the Non-Trivial Zeros
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July 1, 2025

This blog post will be related to developing and refreshing my knowledge about the Riemann Zeta function and complex analysis. As I have begun approaching the Riemann Hypothesis problem, I had two main thoughts for ascertaining a solution:

- My first idea was as follows. Given my previous post about the introduction to the Riemann Hypothesis, I highlighted that a study has been done verifying that the first 10 trillion non-trivial zeros of the Riemann Zeta function have been shown numerically to have real-part of $\sigma_0 = \frac{1}{2}$. Given this notorious statistic, it seems worthwhile to search for a proof where assuming that the Riemann Hypothesis is false (i.e. that there exists a $\sigma_0 \neq \frac{1}{2}$) leads to a contradiction. A direct way to approach this strategy is to use the functional equation - recall:
$$\zeta\left(s\right) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma\left(1-s\right) \zeta\left(1-s\right) \tag{1}$$
Given the functional equation, if the Riemann Zeta function has a non-trivial zero $s_0 = \sigma_0 + it_0$, where $0 < \sigma_0 < 1$, then it must also have a non-trivial zero at $1 - s_0 = 1 - \sigma_0 - it_0$ (since the gamma and sine terms have no zeros when $0 < \sigma_0 < 1$). In fact, $\Gamma\left(1-s\right)$ has no zeros over the entire complex plane, and $\sin\left(\frac{\pi s}{2}\right)$ only has zeros on the real line for even integer values of σ and 0 imaginary parts. In any case, if one could show through an alternative formulation of the Riemann Zeta function that a proposed zero s_0 necessarily implies that a *distinct* $1 - s_0$ is not a zero if $\sigma_0 \neq \frac{1}{2}$, then the Riemann Hypothesis would be proven. The reason that the case of $\sigma_0 = \frac{1}{2}$ is "different" is because it represents the case where $s_0 = \left(1 - s_0\right)^*$ (where * corresponds to the complex conjugate), which as we will show in this post means that the s_0 and $1 - s_0$ correspond to the *same* zero.

- My second idea was to in some way relate properties of the integration of the Riemann Zeta function parallel to the imaginary axis to the presence of zeros on this interval. Then, an integral could be computed that would possess information about whether a zero had been encountered on that interval. A particularly interesting candidate function idea would be to use the logarithm function (singular at 0) with the absolute value of $\zeta\left(s\right)$ as its argument, integrating parallel to the imaginary axis. Such complex logarithm with an absolute value as its argument only has poles at inputs of zero (i.e. where the zeros of the Riemann Zeta function reside). This method is something that will be explored in later posts, and seems somewhat less straightforward than my first idea, since it is somehow difficult to discern based only on a single integral value what values (potentially zeros) are being averaged over the interval in the complex plane. A particularly interesting endeavour will be discovering the art of integrating the Riemann Zeta function in the complex plane parallel to the imaginary axis between $0 < \sigma < 1$ at all.

To begin the formal analysis of this problem, it is first necessary to introduce some basic properties of the Riemann Zeta function in the complex plane. I would like to begin by presenting some analysis that I found from a YouTube video given by Andrew Sotomayor which introduced me to the problem of working with the (troublesome) Riemann Zeta function. First, let's begin with the expressing the Riemann Zeta for $\sigma > 1$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{2}$$

We are interested in attaining the analytic continuation of $\zeta(s)$ to $0 < \sigma < 1$. Note that this interval is particularly special as it is the only interval that is not covered by combining Equation 2 and Equation 1. First, we introduce something known as the Dirichlet Eta function, valid for $0 < \sigma < 1$:

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \tag{3}$$

Writing out this function explicitly and manipulating, we find:

$$\begin{aligned}
\eta(s) &= 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \dots \\
\implies \zeta(s) - \eta(s) &= 2\left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s}\right) \\
&= \frac{2}{2^s}\zeta(s) \\
\implies \zeta(s) &= \frac{1}{1-\frac{2}{2^s}}\eta(s) \\
&= \left(\frac{1}{1-\frac{2}{2^s}}\right)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}
\end{aligned} \tag{4}$$

where the last line is valid for $0 < \sigma < 1$. Sotomayor then continues to show a method for finding non-trivial zeros using a graphing calculator. First, from the last line of Equation 4, we can show that:

$$\begin{aligned}
\zeta(s) &= \left(\frac{1}{1-\frac{2}{2^s}}\right)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\sigma}n^{it}} \\
&= \left(\frac{1}{1-\frac{2}{2^s}}\right)\sum_{n=1}^{\infty}(-1)^{n-1}n^{-\sigma}e^{-it\ln(n)} \\
&= \left(\frac{1}{1-\frac{2}{2^s}}\right)\left(\sum_{n=1}^{\infty}(-1)^{n-1}n^{-\sigma}\cos(t\ln(n)) - i\sum_{n=1}^{\infty}(-1)^{n-1}n^{-\sigma}\sin(t\ln(n))\right)
\end{aligned} \tag{5}$$

We arrive at a point I made previously: if $s_0 = \sigma_0 + it_0$ is a zero as in the last line of Equation 5, then s^* is also a zero. Furthermore, non-trivial zeros in the complex plane are symmetrical about the real axis; thus, for $\sigma = \frac{1}{2}$ (where all the zeros have been found), we have shown that s_0 and $1 - s_0$ correspond to the same non-trivial zero in the complex plane - the $1 - s_0$ zero is simply reflected along the real number line. However, for $\sigma \neq \frac{1}{2}$, it is clear that s_0 and $1 - s_0$ correspond to two different zeros in the complex plane. In this way, it may be possible to show that if the zero at s_0 exists, then the zero at $1 - s_0$ does not exist, a contradiction of the Equation 1. This represents the substance of my first approach to solving this problem - it will be the topic of my next post. This concludes this blog post.