Title: Prime Number Product of Riemann Zeta Function
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In this blog post, we will prove more identities of the Riemann Zeta function. In particular, we will prove that for $s = \sigma + it$ and $\sigma > 1$:

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} \tag{1}$$

This identity was due to Euler. To prove this relation, we will follow similar reasoning to the proof of the analytic continuation of $\zeta(\sigma + it)$ to $\sigma > 0$ (but remember, this identity does only apply for $\sigma > 1$). First:

$$\begin{aligned} \zeta(s) &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} \dots \\ \implies \frac{1}{2^s}\zeta(s) &= \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} \dots \\ \implies \left(1 - \frac{1}{2^s}\right)\zeta(s) &= 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} \dots \\ \implies \frac{1}{3^s}\left(1 - \frac{1}{2^s}\right)\zeta(s) &= \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \frac{1}{21^s} + \frac{1}{27^s} \dots \\ \implies \left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s) &= 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \frac{1}{13^s} + \frac{1}{17^s} \dots \end{aligned} \tag{2}$$

As can be seen, as more factors of $\left(1 - \frac{1}{n^s}\right)$ are added multiplying $\zeta(s)$, the series of terms that remain are only ones which do not have divisors of any of the n that are present multiplying ζ (as in, in the last line of Equation 2, we do not have any denominators that can be divided without remainder by 2 or 3). Now, since the fundamental theorem of arithmetic says that any integer greater than 1 can be represented as a product of prime numbers, this means that the infinite product of prime numbers all multiplying $\zeta(s)$ should leave no numbers left on the right side except the 1. We obtain:

$$\begin{aligned} \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)\zeta(s) &= 1 \\ \implies \zeta(s) &= \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} \end{aligned} \tag{3}$$

Thus, Equation 3 is proven. It is interesting to consider if there is an analogous prime number identity for the analytic continuation of $\zeta(\sigma + it)$ to $\sigma > 0$. This query is motivated by the philosophical benefit of seeing proofs - if one knows how an answer was achieved, they can better reason about how other problems can be tackled.

This proof will be most straightforward to demonstrate by analysing the series resulting from multiplying by various prime number expressions. For instance:

$$\begin{aligned}
\left(1-\frac{1}{2^s}\right)\eta\left(s\right) &= 1-\frac{2}{2^s}+\frac{1}{3^s}+\frac{1}{5^s}-\frac{2}{6^s}+\frac{1}{7^s}+\frac{1}{9^s}-\frac{2}{10^s}+\frac{1}{11^s}+\dots \\
\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\eta\left(s\right) &= 1-\frac{2}{2^s}+\frac{1}{5^s}+\frac{1}{7^s}-\frac{2}{10^s}+\frac{1}{11^s}+\frac{1}{13^s}-\frac{2}{14^s}+\frac{1}{17^s}\dots \\
\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\eta\left(s\right) &= 1-\frac{2}{2^s}+\frac{1}{7^s}+\frac{1}{11^s}+\frac{1}{13^s}-\frac{2}{14^s}+\frac{1}{17^s}\dots
\end{aligned} \tag{4}$$

Furthermore, the remaining series on the right side only consists of terms that do not contain any of the odd prime divisors multiplying on the left side (that is, the right side of the third line of Equation 4 does not contain any numbers with denominators that can be divided by 3 or 5). All the odd denominators are added once, while the terms with even denominators are negative and multiplied by 2. Thus, the test for all the odd denominators is whether it can be divided by any odd primes on the left side, while the test for even denominators is the same with the additional condition that the denominator cannot be divisible by 4. From this analysis, it is clear that the alternating nature of the η series is such that additional even integer terms come into influence that are not present in Equation 1. However, one may have an intuition that when the product involving all prime numbers is taken on the left side of Equation 4, perhaps only the $1-\frac{2}{2^s}$ may remain on the right side for $\sigma > 0$. However, this can't be true, because then substitution into the formula $\zeta\left(s\right) = \left(1-\frac{2}{2^s}\right)^{-1}\eta\left(s\right)$ would yield Equation 3 for $\sigma > 0$. But that is only true for $\sigma > 1$. Furthermore, it seems that the current form of the factors on the left side of Equation 4 does not easily simplify the right side - however, what we have observed above is remarkably helpful in building a computing model of the product developing in Equation 4 - see the next paragraph.

The method for computing the series resulting from the product in scientific computing software is one in which the modulo function is quite useful. Consider the second line of Equation 4, where the product of the first two primes has been expressed. For this product expression, the resulting series is given by:

$$\begin{aligned}
\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\eta\left(s\right) = \sum_{n=1}^{\infty}&\lceil\bmod\left(2n-1,3\right)/3\rceil\frac{1}{\left(2n-1\right)^s} \\
&-2\lceil\bmod\left(2n,4\right)/4\rceil\lceil\bmod\left(2n,3\right)/3\rceil\frac{1}{\left(2n\right)^s}
\end{aligned} \tag{5}$$

where the idea behind the ceiling functions is to be 1 when no primes divide the nth term and otherwise be 0. This idea can be put together to give the series formula specified for the product over all prime numbers:

$$\begin{aligned}
\prod_{p\,\text{prime}}^{\infty}\left(1-\frac{1}{p^s}\right)\eta\left(s\right) = \sum_{n=1}^{\infty}\prod_{p_{o1}\,\text{odd prime}}^{\infty}&\lceil\bmod\left(2n-1,p_{o1}\right)/p_{o1}\rceil\frac{1}{\left(2n-1\right)^s}+ \\
&-2\lceil\bmod\left(2n,4\right)/4\rceil\prod_{p_{o2}\,\text{odd prime}}^{\infty}\lceil\bmod\left(2n,p_{o2}\right)/p_{o2}\rceil\frac{1}{\left(2n\right)^s}
\end{aligned} \tag{6}$$

Future blog posts will examine more interesting products involving the Dirichlet η function, as well as a philosophical summary of the idea of ∞. This concludes this blog post.