

Title: On The Rationality Of The Euler-Mascheroni Constant

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In my idea generation post, I alluded to my awareness of the problem of the irrationality of the Euler-Mascheroni constant, known as $\gamma = 0.577215\dots$. It is not yet currently known if this constant can be expressed as a rational number or whether it's digits continue infinitely with no discernible pattern. A common definition of this constant is:

$$\gamma = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k} - \ln(n) \quad (1)$$

We start with the following manipulation, where we purposely write the logarithmic series bound as ∞ without specifying its value relative to n :

$$\begin{aligned} \gamma &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k} - \sum_{k=1}^{\infty} \frac{(1 - \frac{1}{n})^k}{k} \\ &\approx \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k} - \sum_{k=1}^{\infty} \frac{1}{k} + \frac{1}{n} \sum_{k=1}^{\infty} (1) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^{\gamma n} (1) = \gamma \end{aligned} \quad (2)$$

where we see that if we make the approximation in line 2 (which by the way is not valid - will explain soon), then the infinite logarithmic series has an upper bound that goes as γn to correct for the approximation. Though it is tempting to make the Taylor series approximation $(1+x)^\alpha \approx 1 + \alpha x$ for small enough x , this approximation assumes that $\alpha \ll \frac{1}{x}$, which isn't necessarily true for all terms in the sum. In this case, since the exponent k is going to ∞ in the upper bound of the sum, one cannot easily approximate, and must handle the corrections to the series term by term. Another wrinkle that makes me feel somewhat uncomfortable in the above is that we cancelled:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k} - \sum_{k=1}^{\gamma n} \frac{1}{k} \approx 0$$

In fact, after consulting with Mathematica, the following result holds:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k} - \sum_{k=1}^{an} \frac{1}{k} = -\ln(a)$$

which appears consistent with the correction of making the logarithmic series upper bound as γn . Now we carry on:

$$\begin{aligned}
\gamma &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k} - \sum_{k=1}^{\infty} \frac{(1 - \frac{1}{n})^k}{k} \\
&= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k} - \sum_{k=1}^{\infty} \frac{1}{k} \sum_{m=0}^k \binom{k}{m} \left(-\frac{1}{n}\right)^m \\
&= -\ln(a) - \lim_{n \rightarrow \infty} \sum_{k=1}^{an} \sum_{m=1}^k \frac{\binom{k}{m}}{k} \left(-\frac{1}{n}\right)^m \\
&= -\lim_{n \rightarrow \infty} \sum_{m=1}^{an} \left(\frac{\binom{an+1}{m}}{m} - \frac{\binom{an+1}{m}}{an+1} \right) \left(-\frac{1}{n}\right)^m \\
&= -\ln(a) - \lim_{n \rightarrow \infty} \left(\frac{\left(-\frac{1}{n}\right)^{an} - (1+an)^2 F_{32}\left(1, 1, -an; 2, 2; \frac{1}{n}\right)}{n(1+an)} \right) \\
&\quad + \frac{1}{1+an} \left(-\left(-\frac{1}{n}\right)^{1+an} + \frac{\left(\frac{n-1}{n}\right)^{an}}{n} - 1 + \left(\frac{n-1}{n}\right)^{an} \right) \\
&= -\ln(a) + \lim_{n \rightarrow \infty} \left(\frac{F_{32}\left(1, 1, -an; 2, 2; \frac{1}{n}\right)}{n} + a F_{32}\left(1, 1, -an; 2, 2; \frac{1}{n}\right) \right) + \frac{1}{1+an} (e^{-a} - 1) \\
&= -\ln(a) + \lim_{n \rightarrow \infty} a F_{32}\left(1, 1, -an; 2, 2; \frac{1}{n}\right) \\
&= -\ln(a) + \lim_{n \rightarrow \infty} \sum_{k=0}^{an} \frac{a}{(k+1)!(k+1)} \prod_{m=0}^{k-1} \left(\frac{-an+m}{n} \right)
\end{aligned} \tag{3}$$

where we note that the γ value is satisfied for the parameter $a \rightarrow \infty$. That is:

$$\begin{aligned}
\gamma &= \lim_{a \rightarrow \infty} \left(-\ln(a) + a F_{32}\left(1, 1, -an; 2, 2; \frac{1}{n}\right) \right) \\
&= \lim_{a \rightarrow \infty} \left(-\ln(a) + \lim_{n \rightarrow \infty} \sum_{k=0}^{an} \frac{a(-an)_k}{(k+1)!(k+1)} \left(\frac{1}{n}\right)^k \right) \\
&= \lim_{a \rightarrow \infty} \left(-\ln(a) + \lim_{n \rightarrow \infty} \sum_{k=0}^{an} \frac{a}{(k+1)!(k+1)} \prod_{m=0}^{k-1} \left(\frac{-an+m}{n} \right) \right)
\end{aligned} \tag{4}$$

where the following summations were taken from Mathematica:

$$\begin{aligned}
\sum_{k=m}^{an} \frac{\binom{k}{m}}{k} &= \frac{\binom{an+1}{m}}{m} - \frac{\binom{an+1}{m}}{an+1} \\
\sum_{m=1}^{an} \frac{\binom{an+1}{m}}{m} \left(-\frac{1}{n}\right)^m &= \frac{\left(-\frac{1}{n}\right)^{an} - (1+an)^2 F_{32}\left(1, 1, -an; 2, 2; \frac{1}{n}\right)}{n(1+an)} \\
\sum_{m=1}^{an} \frac{\binom{an+1}{m}}{an+1} \left(-\frac{1}{n}\right)^m &= -\left(-\frac{1}{n}\right)^{1+an} + \frac{\left(\frac{n-1}{n}\right)^{an}}{n} - 1 + \left(\frac{n-1}{n}\right)^{an}
\end{aligned}$$

It is immediately clear that the following identity holds:

$$\lim_{a \rightarrow \infty} \lim_{n \rightarrow \infty} aF_{32} \left(1, 1, -an; 2, 2; \frac{1}{n} \right) = \lim_{a \rightarrow \infty} \sum_{k=1}^a \frac{1}{k} \quad (5)$$

However, this does not help significantly in determining if the Euler-Mascheroni constant is rational or irrational. Note that Equation 1 does not imply that n or a are integers, since the same limiting definition holds for:

$$\gamma = \lim_{n \rightarrow \infty} \sum_{k=1}^{\lfloor n \rfloor} \frac{1}{k} - \ln(n)$$

which holds for real n . From the Equation 5, an additional identity (that involves taking the derivative of Equation 4 with respect to a , then evaluating $a \rightarrow \infty$) is true:

$$\lim_{a \rightarrow \infty} \lim_{n \rightarrow \infty} \left(F_{32} \left(1, 1, -an; 2, 2; \frac{1}{n} \right) - anF_{32}^{(0,0,1;0,0;0)} \left(1, 1, -an; 2, 2; \frac{1}{n} \right) \right) \sim \lim_{a \rightarrow \infty} \frac{1}{a} = 0 \quad (6)$$

An interesting philosophical perspective is that as formulated, this constant $a \rightarrow \infty$ entails that the upper bound of the logarithmic series is infinitely more vast than the upper bound of the harmonic number series (first term in Equation 1). In fact, this is a very foreseeable conclusion insofar as we trust the logical principle that the interpretation of the logarithmic series' infinite upper bound should not be subjective based on the problem in which it is implemented. This concludes this blog post.