Title: Infinite Summations Involving the Harmonic Number
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This blog post will summarize the method used by De-Yin Zheng in 2007 to find solutions to infinite series involving the Harmonic Numbers. This paper can be found through Google Scholar on the worldwide web. This post is directly related to the post discussing the integration of a special class of logarithmic functions, where it was found that the solution is attainable in terms of a single infinite series involving powers of the generalized harmonic numbers. Though a generalized solution to the integral for all positive integers a and b will still be elusive from this post, this post provides the general method that, with enough mathematical accounting, would make the solution tenable.

To begin, we will present a result proven by Gauss:

$$ {}_2F_1\left(x,y;1-z;1\right) = \Gamma\begin{bmatrix}1-z, 1-x-y-z\\ 1-x-z, 1-y-z\end{bmatrix} \tag{1} $$

where ${}_2F_1\left(x,y;1-z;1\right)$ is Gauss' hypergeometric function, and $\Gamma\left(a_1,\ldots,a_n;b_1\ldots,b_m\right) = \frac{\Gamma(a_1)\ldots\Gamma(a_n)}{\Gamma(b_1)\ldots\Gamma(b_m)}$. Now we will introduce some series identities:

$$ \begin{aligned} \Gamma\left(1-z\right) &= e^{\sum_{k=1}^{\infty}\left(\frac{\sigma_k}{k}z^k\right)} \\ \prod_{k=1}^{n}\left(1+\frac{x}{k}\right) &= \sum_{k=0}^{n}\left(\frac{\begin{bmatrix}n+1\\k+1\end{bmatrix}}{n!}x^k\right) = \sum_{k=0}^{n}\left(\frac{H_{k+1}^{\sum_{-}^{n+1}}}{k!}x^k\right) \\ \prod_{k=1}^{n}\left(1-\frac{x}{k}\right)^{-1} &= \sum_{k=0}^{n}\left(\frac{H_{k+1}^{\sum_{+}^{n+1}}}{k!}x^k\right) \end{aligned} \tag{2} $$

where $\sigma_k = \zeta\left(k\right)$ (where $\sigma_1 = \gamma$, the Euler-Mascheroni constant), and where the second identity was discussed in the post discussing the generator of explicit expressions of the Stirling Numbers of the First Kind post. The functions $H_{k+1}^{\sum_{+}^{n+1}}$ and $H_{k+1}^{\sum_{-}^{n+1}}$ are the explicit expressions of the Stirling Numbers of the First Kind in terms of the Harmonic Numbers. For example, for $k = 2$, $\begin{bmatrix}n+1\\3\end{bmatrix} = \frac{n!}{2!}H_3^{\sum_{-}^{n+1}}$, where $H_3^{\sum_{-}^{n+1}} = H_n^2 - H_n^{(2)}$. Subsequently, $H_3^{\sum_{+}^{n+1}} = H_n^2 + H_n^{(2)}$. Furthermore, $H_{k+1}^{\sum_{-}^{n+1}}$ and $H_{k+1}^{\sum_{+}^{n+1}}$ are similar except that $H_{k+1}^{\sum_{+}^{n+1}}$ is the series of powers of the Harmonic Numbers *without* any minus signs, while $H_{k+1}^{\sum_{-}^{n+1}}$ is the series with minus and plus signs that is characteristic of the Stirling Numbers of the First Kind. With these identities defined, we can now proceed with the multivariate relation defined in Equation 1. First, the left side of Equation 1:

$$\begin{aligned}
{}_2F_1\left(x, y; 1-z; 1\right) &= \sum_{n=0}^{\infty} \frac{(x)_n\,(y)_n}{(1-z)_n} \frac{1}{n!} \\
&= 1 + \sum_{n=1}^{\infty} \frac{(x)_n\,(y)_n}{(1-z)_n} \frac{1}{n!} \\
&= 1 + xy \sum_{n=1}^{\infty} \frac{(1+x)_{n-1}\,(1+y)_{n-1}}{(1-z)_n} \frac{1}{n!} \\
&= 1 + xy \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{n-1}(i+x)(i+y)}{\prod_{j=1}^{n}(j-z)} \frac{1}{n!} \\
&= 1 + xy \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{n-1}\left(1+\frac{x}{i}\right)\left(1+\frac{y}{i}\right)}{n^2 \prod_{j=1}^{n}\left(1-\frac{z}{j}\right)}
\end{aligned} \tag{3}$$

where $(x)_n = \prod_{i=1}^{n}(x+i-1)$ is the rising Pochhammer Symbol. Implementing the last two lines of Equation 2, we obtain a multivariate series in terms of x, y and z. Using the first line of Equation 2, we can write the right side of Equation 1 in terms of a multivariate series as well:

$$\begin{aligned}
\Gamma\begin{bmatrix} 1-z, 1-x-y-z \\ 1-x-z, 1-y-z \end{bmatrix} &= e^{\sum_{k=1}^{\infty} \frac{\sigma_k}{k}\left(z^k+(x+y+z)^k-(x+z)^k-(y+z)^k\right)} \\
&= e^{\sigma_2 xy+\sigma_3 xy(x+y+2z)+\sigma_4 xy\left(x^2+y^2+\frac{3}{2}xy+3xz+3yz+3z^2\right)+\ldots}
\end{aligned} \tag{4}$$

Now, applying the Maclaurin series ($e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$) to Equation 4, and then setting it equal to the series from Equation 3 (applying the last two lines of Equation 2) via Equation 1 leads to many identities such as (by setting the coefficients of the powers of x, y, and z equal in the multivariate series equality), including:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n}{n^2} &= 2\zeta(3) \\
\sum_{n=1}^{\infty} \frac{H_{n-1}H_n}{n^2} &= 3\zeta(4) \\
\sum_{n=1}^{\infty} \frac{H_n^{(2)}-H_n^2}{n^3} &= \frac{2\pi^2}{3}\zeta(3) - 8\zeta(5)
\end{aligned} \tag{5}$$

Through other similar summation theorems such as the one in Equation 1, such as that by Dougall and Dixon (see Zheng 2007), we obtain other results, such as:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n^3}{n^2} &= \frac{\pi^2}{6}\zeta(3) + 10\zeta(5) \\
\sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^3} &= \frac{\pi^6}{1890} + \frac{\zeta(3)^2}{2}
\end{aligned} \tag{6}$$

Zheng 2007 lists many other identities from these multivariate series expansions. This concludes this blog post.