Title: Series Arising from Fourier Representation
Author: Josh Myers

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Last post, we considered the Fourier Coefficient defined as:

$$c_n = \frac{1}{L}\int_0^L g\left(x'\right) e^{-2\pi in\frac{x'}{L}}\mathrm{d}x' \tag{1}$$

Through methods discussed in the last post ("PDE Fourier Series: Taylor Series"), we derived the series representation for c_n, $n\frac{\partial c_n}{\partial n}$, and $L\frac{\partial c_n}{\partial L}$ and related them to the foundational PDE. The series representations derived are:

$$\begin{aligned} c_n &= \sum_{m=0}^{\infty}\left(\frac{L}{2\pi in}\right)^m \frac{g^{(m)}\left(0\right)}{m!}\frac{\gamma\left(m+1,2\pi in\right)}{2\pi in} \\ n\frac{\partial c_n}{\partial n} &= -\sum_{m=0}^{\infty}\left(\frac{L}{2\pi in}\right)^m \frac{g^{(m)}\left(0\right)}{m!}\frac{\gamma\left(m+2,2\pi in\right)}{2\pi in} \\ L\frac{\partial c_n}{\partial L} &= \sum_{m=0}^{\infty}\left(\frac{L}{2\pi in}\right)^{m+1} \frac{g^{(m+1)}\left(0\right)}{m!}\frac{\gamma\left(m+2,2\pi in\right)}{2\pi in} \end{aligned} \tag{2}$$

In this post, we will state the equivalent series representations for the altered Fourier coefficient definition:

$$c_n = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} g\left(x'\right) e^{-2\pi in\frac{x'}{L}}\mathrm{d}x' \tag{3}$$

The series representations analogous to Equation 2 for the updated Equation 3 are:

$$\begin{aligned} c_n &= \sum_{m=0}^{\infty}\left(\frac{L}{2\pi in}\right)^m \frac{g^{(m)}\left(0\right)}{m!}\frac{\gamma\left(m+1,\pi in\right)-\gamma\left(m+1,-\pi in\right)}{2\pi in} \\ n\frac{\partial c_n}{\partial n} &= -\sum_{m=0}^{\infty}\left(\frac{L}{2\pi in}\right)^m \frac{g^{(m)}\left(0\right)}{m!}\frac{\gamma\left(m+2,\pi in\right)-\gamma\left(m+2,-\pi in\right)}{2\pi in} \\ L\frac{\partial c_n}{\partial L} &= \sum_{m=0}^{\infty}\left(\frac{L}{2\pi in}\right)^{m+1} \frac{g^{(m+1)}\left(0\right)}{m!}\frac{\gamma\left(m+2,\pi in\right)-\gamma\left(m+2,-\pi in\right)}{2\pi in} \end{aligned} \tag{4}$$

As given in the last post, the lower incomplete gamma function is given by:

$$\gamma\left(s+1,z\right) = \int_0^z x^s e^{-x}\mathrm{d}x = -\sum_{q=0}^{s-1} z^{s-q}\frac{s!}{(s-q)!}e^{-z} - s!\left(e^{-z}-1\right) \tag{5}$$

Substituting Equation 5 into the top line of Equation 4, we find the general solution:

$$\begin{aligned} c_n &= \sum_{m=0}^{\infty}\left(\sum_{q=0}^{m-1}\left(\left(\frac{L}{2\pi in}\right)^m \frac{g^{(m)}(0)}{m!}\frac{1}{2\pi in}\left(\left(-(i\pi n)^{m-q}+(-i\pi n)^{m-q}\right)\frac{m!}{(m-q)!}(-1)^n\right)\right)\right) \\ &= \sum_{m=0}^{\infty}\sum_{q=0}^{m-1}\left(\frac{L}{2}\right)^m\left(\frac{1}{i\pi n}\right)^{q+1}\frac{g^{(m)}(0)}{(m-q)!}\frac{\left(1-(-1)^{m-q}\right)}{2}(-1)^{n+1} \end{aligned} \tag{6}$$

Now this formula can be used for analysis. Assume that $g(x) = x^s$, where s is a positive integer. In this case, $g^{(m)}(0) = m!\,\delta_{ms}$. Then, c_n is given as:

$$c_n = \sum_{q=0}^{s-1}\left(\frac{L}{2}\right)^s\left(\frac{1}{i\pi n}\right)^{q+1}\frac{s!}{(s-q)!}\frac{\left(1-(-1)^{s-q}\right)}{2}(-1)^{n+1} \tag{7}$$

This solution for c_n implies that every even or odd power greater than 0 and less than or equal to s (depending on if s is even or odd, respectively) will have a reciprocal power of n associated. This means that the resulting Fourier Series (once $x = 0$ and $L = 1$ are taken) will be in terms of Riemann Zeta functions in the integers. This general c_n can now be substituted into the formula for the Fourier Series, setting $x = 0$ and $L = 1$, and we get:

$$\begin{aligned} &0 = \frac{1}{s+1}\left(\frac{1}{2}\right)^s + \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{s-1}(-1)^{n+1}s!\sum_{q=0}^{s-1}\left(\frac{1}{i\pi n}\right)^{q+1}\frac{1}{(s-q)!}\frac{1-(-1)^{s-q}}{2}\ \text{for } s \text{ even, and} \\ &0 = 0 \text{ for } s \text{ odd} \end{aligned} \tag{8}$$

Now we zoom in on s even - switch the order of summation of n and q. Note that the first term in the Equation 8 is the c_0 term from the Fourier Series. We then get our Riemann Zeta function:

$$0 = \frac{1}{s+1}\left(\frac{1}{2}\right)^s + \left(\frac{1}{2}\right)^{s-1}s!\sum_{q=0}^{s-1}\left(\frac{1}{i\pi}\right)^{q+1}\frac{1}{(s-q)!}\frac{1-(-1)^{s-q}}{2}\left(1-\frac{2}{2^{q+1}}\right)\zeta(q+1) \tag{9}$$

This relation means that we can solve for even order Riemann Zeta's in terms of lower even order Riemann Zeta's on the positive integers. In particular, it is evident that:

$$\zeta(s) = -\frac{(2\pi i)^s}{2s!\left(1-\frac{2}{2^s}\right)}\left(\frac{1}{s+1}\left(\frac{1}{2}\right)^s + \left(\frac{1}{2}\right)^{s-1}s!\sum_{q=0}^{s-3}\left(\frac{1}{i\pi}\right)^{q+1}\frac{1}{(s-q)!}\frac{1-(-1)^{s-q}}{2}\left(1-\frac{2}{2^{q+1}}\right)\zeta(q+1)\right) \tag{10}$$

Indeed, this is one way to solve the Basel Problem, a historically important problem first solved by Euler:

$$\zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = -\frac{(2\pi i)^2}{2}\left(\frac{1}{12}\right) = \frac{\pi^2}{6} \tag{11}$$

I suspect that Equation 10 is not new, and that there are many more insights to be found from studying Fourier Series as methods to solutions of a general series. I will show that the known solution for $\zeta(4)$ is also attained from Equation 10:

$$\zeta(4) = \sum_{n=1}^{\infty}\frac{1}{n^4} = -\frac{(2\pi i)^4}{4!2\left(\frac{7}{8}\right)}\left(\frac{1}{80}-\frac{1}{24}\right) = \frac{\pi^4}{90} \tag{12}$$

This concludes this blog post.