Title: Examining Fractional Calculus
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July 1, 2025

Recall the fractional integral formula:

$$(J^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,\mathrm{d}t \tag{1}$$

where $J^{\alpha} J^{\beta} = J^{\alpha+\beta}$. This can be seen by the following calculation:

$$\begin{aligned}
J^{\alpha} J^{\beta} &= \frac{1}{\Gamma(\alpha)} \int_0^x \mathrm{d}u\,(x-u)^{\alpha-1} \frac{1}{\Gamma(\beta)} \int_0^u \mathrm{d}t\,(u-t)^{\beta-1} f(t) \\
&= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_0^x \mathrm{d}t\, f(t) \int_t^x \mathrm{d}u\,(x-u)^{\alpha-1} (u-t)^{\beta-1} \\
&= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_0^x \mathrm{d}t\, f(t) \int_0^{x-t} \mathrm{d}z\ (x-t-z)^{\alpha-1} z^{\beta-1} \\
&= \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\beta)} \int_0^x \mathrm{d}t\ (x-t)^{\alpha+\beta-1} f(t) \int_0^1 \mathrm{d}w\ (1-w)^{\alpha-1} w^{\beta-1} \\
&= \frac{1}{\Gamma(\alpha+\beta)} \int_0^x \mathrm{d}t\ (x-t)^{\alpha+\beta-1} f(t) \\
&= J^{\alpha+\beta}
\end{aligned} \tag{2}$$

where the fourth to the fifth line is true because the complete beta function is $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$. Now that we've verified this identity, we can move forward with the idea that the fractional derivative is the symbolic “inverse” operation of the fractional integral - that is, $(D^{\alpha} J^{\alpha} f)(x) = f(x)$. This identity can be useful for solving for $f(x)$ given $g_{\alpha}(x) = (J^{\alpha} f)(x)$. In the case where k is a positive integer and $0 \leq \alpha \leq 1$, we can find for $g_{k,\alpha}(x) = \left(J^{k+\alpha} f\right)(x)$ for a specific value of α:

$$f(x) = \left(D^{k+1} J^{1-\alpha} g_{k,\alpha}\right)(x) \tag{3}$$

which is necessary since D^{k+1} only exists for integer k (the standard definition of the derivative) and $J^{1-\alpha}$ exists for any $\alpha \leq 1$. Note that this formula is profound because it means that after $f(x)$ is calculated, all α and k dependence will be eliminated.

One of the exotic ideas that one can consider with fractional calculus is averaging over the integrated function space. For this idea, we consider the following calculation:

$$I = \int_0^1 \mathrm{d}\alpha \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \tag{4}$$

To carry out this calculation, we make use of the following identity from Wolfram Mathworld Gamma function page:

$$\frac{1}{\Gamma(\alpha)} = \alpha e^{\gamma\alpha - \sum_{k=2}^{\infty} \frac{(-1)^k \zeta(k) \alpha^k}{k}} \tag{5}$$

where γ is the Euler-Mascheroni constant. This relation can be used as follows:

$$\begin{aligned} I &= \int_0^1 \mathrm{d}\alpha \, (x-t)^{\alpha-1} \, \alpha e^{\gamma\alpha - \sum_{k=2}^{\infty} \frac{(-1)^k \zeta(k) \alpha^k}{k}} \\ &= \frac{1}{x-t} \int_0^1 \mathrm{d}\alpha \, \alpha e^{\alpha(\gamma + \ln(x-t)) - \sum_{k=2}^{\infty} \sum_{n=1}^{\infty} \frac{\left(-\frac{\alpha}{n}\right)^k}{k}} \\ &= \frac{1}{x-t} \int_0^1 \mathrm{d}\alpha \, \alpha e^{\alpha(\gamma + \ln(x-t)) - \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{\left(-\frac{\alpha}{n}\right)^k}{k} - \sum_{n=1}^{\infty} \frac{\alpha}{n}} \\ &= \frac{1}{x-t} \int_0^1 \mathrm{d}\alpha \, \alpha e^{\alpha(\gamma + \ln(x-t) - \lim_{n\to\infty} H_n) + \sum_{n=1}^{\infty} \ln\left(1+\frac{\alpha}{n}\right)} \\ &= \frac{1}{x-t} \int_0^1 \mathrm{d}\alpha \, \alpha e^{-\alpha(\lim_{n\to\infty} H_n - \gamma - \ln(x-t))} \lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{\alpha}{k}\right) \end{aligned} \tag{6}$$

From previous posts on this blog, we have found that:

$$\prod_{k=1}^{n} \left(1 + \frac{\alpha}{k}\right) = \sum_{k=0}^{n} \frac{\left[{n+1 \atop k+1}\right]}{n!} \alpha^k \tag{7}$$

where $\left[{n+1 \atop k+1}\right]$ is the Stirling Numbers of the first kind. Substituting Equation 7 into Equation 6, we get:

$$\begin{aligned} I &= \frac{1}{x-t} \lim_{n\to\infty} \sum_{k=0}^{n} \frac{\left[{n+1 \atop k+1}\right]}{n!} \int_0^1 \mathrm{d}\alpha \, \alpha^{k+1} e^{-\alpha(\lim_{n\to\infty} H_n - \gamma - \ln(x-t))} \\ &= \frac{1}{x-t} \lim_{n\to\infty} \sum_{k=0}^{n} \frac{\left[{n+1 \atop k+1}\right]}{n!} \frac{\Gamma(k+2) - \Gamma(k+2, \lim_{n\to\infty} H_n - \gamma - \ln(x-t))}{(\lim_{n\to\infty} H_n - \gamma - \ln(x-t))^{k+2}} \end{aligned} \tag{8}$$

where the $\Gamma(s,z) = \int_z^\infty \mathrm{d}t \, t^{s-1} e^{-t}$ is the incomplete gamma function. I have checked this formula numerically. The interesting part of this expression is that it acts as a fundamental form for averaging over an integrated function space. In completeness, where $G(x) = \int_0^1 \mathrm{d}\alpha \, g_\alpha(x)$:

$$G(x) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{\left[{n+1 \atop k+1}\right]}{n!} \int_0^x \mathrm{d}t \, \frac{\Gamma(k+2) - \Gamma(k+2, \lim_{n\to\infty} H_n - \gamma - \ln(x-t))}{(x-t)(\lim_{n\to\infty} H_n - \gamma - \ln(x-t))^{k+2}} f(t) \tag{9}$$

This result inspires many creative thoughts. Can we begin to do statistics (such as finding the variance and other statistical moments) of an integrated function space? What can this tell us about the integrated function space? The expected value of the integrated function space is:

$$\mathbb{E}_0^1[f(x)] = G(x) = \int_0^1 \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t) \, \mathrm{d}t \, \mathrm{d}\alpha \tag{10}$$

We will follow up on this statistical outlook in future posts. This concludes this blog post.