

Title: Circular Intersection Geometry
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In this post, we will analyze the average area contained in the intersection of two circles of equal radius displaced from each other in space. Consider one circle centre to be at the origin, while the other circle's centre is distance d from the origin, where $0 \leq d \leq 2r$ such that the two circles intersect at one or more points. The foundational question this post is centred around can then be posed: what area is contained in the intersecting portions of the two circles? First, we consider symmetries in the geometry - see the plot in Figure 1.

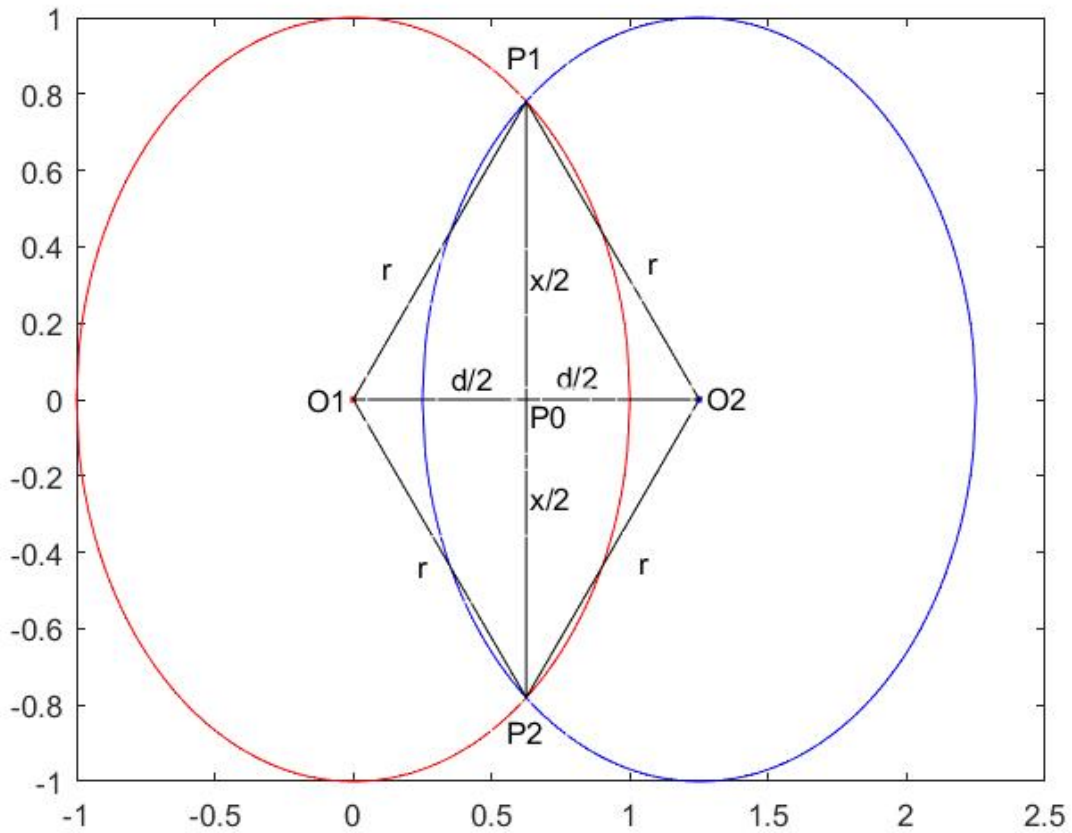


Figure 1: Geometry we'd like to analyze. The radii of the circles are both 1 in this plot.

So we have two circles and one similar triangle (oriented in 4 different ways) to analyze to find the area between red circle and the blue circle (enclosed on the left of P_0 and the right of P_0 , within the circular intersection between P_1 and P_2). First, we write the circular “pizza” area for one of the circles as the following, where we define the angle θ to correspond to the angle between the lines connecting the circle centres (either O_1 or O_2) to P_1 and P_2 :

$$\begin{aligned} A_{\text{pizza}} &= \frac{\theta}{2\pi} A_{\text{circle}} \\ &= \frac{\theta}{2} r^2 \end{aligned} \quad (1)$$

Then, we find the area of one of the four similar right triangles as:

$$\begin{aligned} 2A_{\text{triangle}} &= \left(\frac{d}{2}\right) \left(\frac{x}{2}\right) \\ &= \frac{1}{4} dx \end{aligned} \quad (2)$$

The goal is to find solutions for Equation 1 and Equation 2 only in terms of r and d , such that we can take $A_{\text{pizza crust}} = A_{\text{pizza}} - 2A_{\text{triangle}}$ and then multiply that by 2 to find the area contained by the intersecting circles. To do this, from triangular trigonometry and Pythagorean Theorem, it is clear that:

$$\begin{aligned} \frac{\theta}{2} &= \arccos\left(\frac{d}{2r}\right) \\ x &= 2r\sqrt{1 - \left(\frac{d}{2r}\right)^2} \end{aligned} \quad (3)$$

From this formula, we can now find the area of the “pizza crust” that corresponds to one half of the circle area contained in the intersection area between P_1 and P_2 - $A_{\text{POI}} = 2A_{\text{pizza crust}}$:

$$\begin{aligned} A_{\text{POI}} &= 2 \left(r^2 \arccos\left(\frac{d}{2r}\right) - \frac{1}{2} rd \sqrt{1 - \left(\frac{d}{2r}\right)^2} \right) \\ &= 2r^2 \left(\arccos\left(\frac{d}{2r}\right) - \frac{d}{2r} \sqrt{1 - \left(\frac{d}{2r}\right)^2} \right) \end{aligned} \quad (4)$$

As we can see, for two circles with radius r , the intersection area can be fully expressed in terms of $\frac{d}{2r}$. Now, we can get creative. For two circles with fixed radius r , we can vary the value of $\frac{d}{2r}$ from 0 to 1 and find the average A_{POI} for $0 \leq d \leq 2r$. This integration is shown below:

$$\begin{aligned} A_{\text{POI,av}} &= 2r^2 \int_0^1 du \left(\arccos(u) - u\sqrt{1 - u^2} \right) \\ &= \frac{4}{3} r^2 \end{aligned} \quad (5)$$

This idea can be extended to account for alternative parameterizations of $d = z^s$, $s > 0$, such as:

$$A_{\text{POI,av,s}} = \frac{1}{2} (2r)^{2-\frac{1}{s}} \int_0^{(2r)^{\frac{1}{s}}} dz \left(\arccos\left(\frac{z^s}{2r}\right) - \frac{z^s}{2r} \sqrt{1 - \left(\frac{z^s}{2r}\right)^2} \right) \quad (6)$$

This parameterization allows us to find some interesting limiting expressions in terms of the value of s . For example:

$$\begin{aligned}
A_{\text{POI,av,lim}} &= \lim_{s \rightarrow \infty} \left(\frac{1}{2} (2r)^{2-\frac{1}{s}} \int_0^{(2r)^{\frac{1}{s}}} dz \left(\arccos \left(\frac{z^s}{2r} \right) - \frac{z^s}{2r} \sqrt{1 - \left(\frac{z^s}{2r} \right)^2} \right) \right) = \pi r^2 \\
\Rightarrow \pi &= \lim_{s \rightarrow \infty} \left(2 \int_0^{(2r)^{\frac{1}{s}}} dz \left(\arccos \left(\frac{z^s}{2r} \right) - \frac{z^s}{2r} \sqrt{1 - \left(\frac{z^s}{2r} \right)^2} \right) \right) \\
&= \lim_{s \rightarrow \infty} \left(2 \int_0^1 dz \left(\arccos \left(\frac{z^s}{2r} \right) - \frac{z^s}{2r} \sqrt{1 - \left(\frac{z^s}{2r} \right)^2} \right) \right)
\end{aligned} \tag{7}$$

This relation is particularly interesting, as any particular value depends on r (for finite r), but the left side is independent of r . We can carry out similar analysis to other posts on this blog, where we can take derivatives with respect to “dummy” variables while setting the derivative equal to 0 for all values of r . In this case, there are an infinite number of integral expressions for the above that give you the value of π . This concept all comes from the geometric intuition of the parameterization. It also turns out that this relation comes trivially out of evaluating the integral for any r - first note that:

$$\begin{aligned}
\partial_x \arccos(x) &= -\frac{1}{\sqrt{1-x^2}} \\
\partial_x \arcsin(x) &= \frac{1}{\sqrt{1-x^2}}
\end{aligned} \tag{8}$$

These derivatives imply the following relationship:

$$\begin{aligned}
\arccos(x) + \arcsin(x) &= C_0 \\
\Rightarrow C_0 &= \frac{\pi}{2} \\
\Rightarrow \arccos(x) &= \frac{\pi}{2} - \arcsin(x)
\end{aligned} \tag{9}$$

Now, we note the following series identities:

$$\begin{aligned}
\arcsin(x) &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} (n!)^2} \frac{x^{2n+1}}{2n+1} \\
x\sqrt{1-x^2} &= x - \frac{x^3}{2} - \sum_{n=0}^{\infty} \frac{(2n+1)!}{2^{2n} (n!)^2} \frac{x^{2n+5}}{(n+1)(n+2)}
\end{aligned} \tag{10}$$

Now substituting these series into Equation 7, eliminating the first two terms in the second line of Equation 9 as they integrate out, we find:

$$\pi = \pi + \lim_{s \rightarrow \infty} \int_0^1 dz \left(-2 \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n} (n!)^2} \frac{\left(\frac{z^s}{2r}\right)^{2n+1}}{2n+1} + 2 \sum_{n=0}^{\infty} \frac{(2n+1)!}{2^{2n} (n!)^2} \frac{\left(\frac{z^s}{2r}\right)^{2n+5}}{(n+1)(n+2)} \right) \tag{11}$$

All terms integrate out for non-limiting values of r , leaving the result as just π . In limiting cases of r , such as $r \rightarrow \frac{1}{s}$, we find:

$$A_{\text{POI,av,lim2}} = \lim_{s \rightarrow \infty} \left(\frac{1}{2} \left(\frac{2}{s} \right)^{2-\frac{1}{s}} \int_0^{\left(\frac{2}{s}\right)^{\frac{1}{s}}} dz \left(\arccos \left(\frac{sz^s}{2} \right) - \frac{sz^s}{2} \sqrt{1 - \left(\frac{sz^s}{2} \right)^2} \right) \right) \tag{12}$$

Geometric intuition tells us that this limit should go to 0, since the radius of the equal circles is going to 0, so the area contained in the circular intersection goes to 0 for the parameterization. However, if we divide both sides by πr^2 , we get a normalized area with a limit that is slightly more interesting:

$$\begin{aligned}
A_{\text{POI,av,lim2,norm}} &= \lim_{s \rightarrow \infty} \left(\frac{2}{\pi} \int_0^{\left(\frac{2}{s}\right)^{\frac{1}{s}}} dz \left(\arccos \left(\frac{sz^s}{2} \right) - \frac{sz^s}{2} \sqrt{1 - \left(\frac{sz^s}{2} \right)^2} \right) \right) \\
&= \lim_{s \rightarrow \infty} \frac{\left(2 \int_0^{\left(\frac{2}{s}\right)^{\frac{1}{s}}} dz \left(\arccos \left(\frac{sz^s}{2} \right) - \frac{sz^s}{2} \sqrt{1 - \left(\frac{sz^s}{2} \right)^2} \right) \right)}{\left(2 \int_0^{\left(\frac{2}{s}\right)^{\frac{1}{s}}} dz \left(\arccos \left(\frac{sz^s}{2} \right) - \frac{sz^s}{2} \sqrt{1 - \left(\frac{sz^s}{2} \right)^2} \right) \right)} \\
&= 1
\end{aligned} \tag{13}$$

as expected, as the parameterization is such that the average intersecting area is localized at $d = 0$. This concludes this blog post.