Title: Euler Maclaurin Summation Formula
Author: Josh Myers

July 1, 2025

This post will cover the Euler-Maclaurin summation formula that describes the connection between a series on the integers and the associated integral expression. The proof to follow was taken from Wikipedia. Consider the first the following general integral expression:

$$I\left(m,n\right)=\int_{m}^{n}f\left(x\right)\mathrm{d}x \tag{1}$$

Now consider integrating by parts, where we introduce the Bernoulli Polynomials $B_k\left(x\right)$, $k>0$ belongs to the integers. The first 4 Bernoulli Polynomials (taken from the internet) are:

$$\begin{aligned}
B_0\left(x\right)&=1\\
B_1\left(x\right)&=x-\frac{1}{2}\\
B_2\left(x\right)&=x^2-x+\frac{1}{6}\\
B_3\left(x\right)&=x^3-\frac{3}{2}x^2+\frac{1}{2}x
\end{aligned} \tag{2}$$

The Bernoulli Polynomials have the property that for all k other than $k=1$, $B_k\left(1\right)=B_k\left(0\right)$. Additionally, for all odd k other than $k=1$, $B_k\left(0\right)=0$. Additionally, for all k, Bernoulli Polynomials have the following derivative property:

$$\partial_x B_k\left(x\right)=kB_{k-1}\left(x\right) \tag{3}$$

For all k other than $k=1$, the Bernoulli Numbers B_k are defined by:

$$B_k=B_k\left(1\right)=B_k\left(0\right) \tag{4}$$

The Bernoulli Polynomials are convenient functions to integrate by parts with because if the integral is from 0 to 1, then $B_k\left(1\right)=B_k\left(0\right)=B_k$, the Bernoulli number associated with k. As well, integrating by parts involves taking anti-derivatives of the Bernoulli Polynomials, which are given by $\int B_k\left(x\right)\mathrm{d}x=\frac{1}{k+1}B_{k+1}\left(x\right)$. However, this description has an inherent drawback - the series may run from 0 to n. But it still seems possible to have the advantageous properties without the drawbacks of summation / integration domain - more machinery is needed. Consider what Wikipedia calls the "periodic" Bernoulli Polynomials of order k defined on the full domain of the real numbers, with sections with period 1 that have all the properties of the Bernoulli Polynomials. Defined as $P_k\left(x\right)=B_k\left(x-\lfloor x\rfloor\right)$, we can now start to do some integration by parts on a domain of length 1 starting at integer m:

$$\begin{aligned}\int_m^{m+1} f(x)\,\mathrm{d}x &= \left[B_1(x-m)\,f(x)\right]_m^{m+1} - \int_m^{m+1} B_1(x-m)\,f'(x)\,\mathrm{d}x \\ &= \frac{B_1(1)\,f(m+1)}{2} - \frac{B_1(0)\,f(m)}{2} - \int_m^{m+1} P_1(x)\,f'(x)\,\mathrm{d}x \\ \implies \int_m^{m+1} f(x)\,\mathrm{d}x + \int_m^{m+1} P_1(x)\,f'(x)\,\mathrm{d}x &= \frac{f(m+1)+f(m)}{2}\end{aligned} \tag{5}$$

Now we can sum Equation 5 on all integers from m to n, inclusive. This yields the following relation:

$$\begin{aligned}\int_m^n f(x)\,\mathrm{d}x + \int_m^n P_1(x)\,f'(x)\,\mathrm{d}x &= \frac{f(m)}{2} + f(m+1) + \cdots + f(n-1) + \frac{f(n)}{2} \\ \implies \int_m^n f(x)\,\mathrm{d}x + \int_m^n P_1(x)\,f'(x)\,\mathrm{d}x + \frac{f(m)+f(n)}{2} &= \sum_{i=m}^{n} f(i) = S(m,n)\end{aligned} \tag{6}$$

This is the elementary expression of the Euler-Maclaurin summation formula. It represents an exact relation between a function summed over the integers to the same function integrated over the real numbers. Even more interesting is that this expression has infinitely many valid forms by utilizing repeated integration by parts for higher order Bernoulli Polynomials, and through carrying out similar calculation to the above on the second term from the left in Equation 6, we arrive at the general Euler-Maclaurin summation formula for order p:

$$S(m,n) - I(m,n) = \frac{f(m)+f(n)}{2} + \sum_{k=1}^{\lfloor \frac{p}{2} \rfloor} \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(n) - f^{(2k-1)}(m)\right) + R_p \tag{7}$$

where B_{2k} is the $2k$th Bernoulli Number (only every other term survives since $B_{2k+1} = 0$ for $k > 0$), and where R_p is given by:

$$R_p = (-1)^{p+1} \int_m^n \frac{P_p(x)}{p!} f^{(p)}(x)\,\mathrm{d}x \tag{8}$$

Thus, for any positive integer p, there exists a relation between the integration ($I(m,n)$ in Equation 1) and summation ($S(m,n)$ in Equation 6). This detail is particularly useful, as in many cases, there is limiting behaviour in the higher derivatives of $f(x)$ that allows one to treat the summed terms and R_p in the right side of Equation 7 as negligible for sufficiently high p and high n, leading to very useful approximations that cannot be straightforwardly reached through other methods. Such a situation is the case in my previous blog posts, where I showed the work of an author on Math Stackexchange, where the Euler-Maclaurin formula was used to prove an approximation to the hyperfactorial, where $p = 2$, $m = 1$, and $n \gg 1$ were taken. To remind the reader, the solution from the last post was attained as follows (taking $f(x) = x\ln(x)$):

$$\begin{aligned}\ln(H_n) &\approx \int_1^n x\ln(x)\,\mathrm{d}x + \frac{n\ln(n)}{2} + \frac{B_2}{2}\ln(n) \\ \implies H(n) &\approx A n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} e^{-\frac{n^2}{4}}\end{aligned} \tag{9}$$

where $A = 1.28243$ is the Glaisher-Kinkelin Constant. As has been demonstrated here, there is great potential for attaining approximations to summation expressions for $n \gg 1$ (higher upper bound of summation), which provides a tool for studying approximations to infinite series. With this tool, I will investigate in my next post the implications this method may have on approximating the Dirichlet Eta series associated with the Riemann Hypothesis. This concludes this blog post.