Title: Revisiting the Complex Fourier Series
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August 4, 2025

Recall the formulation of the complex Fourier Series:

$$\begin{aligned} g\left(x\right) &= \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n \frac{x}{L}}, \\ c_n &= \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}x' \, g\left(x'\right) e^{-2\pi i n \frac{x'}{L}} \end{aligned} \tag{1}$$

This formulation is independent of L given that $\frac{L}{2} > x$. This means that we ought to be able to integrate the top line of Equation 1 in L without changing the answer. This statement is expressed in the following equation:

$$g\left(x\right) = \frac{1}{L - 2 \mid x \mid} \int_{2|x|}^{L} \mathrm{d}L' \sum_{n=-\infty}^{\infty} \int_{-\frac{L'}{2}}^{\frac{L'}{2}} \mathrm{d}x' \, \frac{g\left(x'\right) e^{2\pi i n \frac{x-x'}{L'}}}{L'} \tag{2}$$

Switching the order of integration in L' and x', we find:

$$g\left(x\right) = \sum_{n=-\infty}^{\infty} \left(\int_{-|x|}^{|x|} \mathrm{d}x' \int_{2|x|}^{L} \mathrm{d}L' + \int_{|x|}^{\frac{L}{2}} \mathrm{d}x' \int_{2x'}^{L} \mathrm{d}L' + \int_{-\frac{L}{2}}^{-|x|} \mathrm{d}x' \int_{-2x'}^{L} \mathrm{d}L' \right) \frac{g\left(x'\right) e^{2\pi i n \frac{x-x'}{L'}}}{\left(L - 2 \mid x \mid\right) L'} \tag{3}$$

Now we are left with integrals of the form:

$$I = \int_{a}^{b} \mathrm{d}t \, \frac{e^{\frac{1}{t}}}{t} \tag{4}$$

Substituting $u = -\frac{1}{t}$, we get:

$$I = -\int_{-\frac{1}{a}}^{-\frac{1}{b}} \mathrm{d}u \, \frac{e^{-u}}{u} = -\mathrm{Ei}\left(\frac{1}{b}\right) + \mathrm{Ei}\left(\frac{1}{a}\right) \tag{5}$$

where Ei is the exponential integral, defined as:

$$\mathrm{Ei}\left(x\right) = -\int_{-x}^{\infty} \mathrm{d}t \, \frac{e^{-t}}{t} \tag{6}$$

where we note that this integral must be understood in terms of the Cauchy principal value since the integrand diverges at $t \to 0$. In particular, the following fact allows the integral to be defined:

$$\begin{aligned} \lim_{t \to 0^-} \frac{e^{-t}}{t} &\to -\infty \\ \lim_{t \to 0^+} \frac{e^{-t}}{t} &\to +\infty \end{aligned} \tag{7}$$

Thus, the integral diverges oppositely on each side of $t \to 0$, such that the diverging parts cancel and only the finite parts contribute to the value of $\mathrm{Ei}(x)$. Proceeding onwards, using Equation 5 to compute Equation 3, we get:

$$
\begin{aligned}
g(x) = \sum_{n=-\infty}^{\infty} & \int_{-|x|}^{|x|} \mathrm{d}x' \frac{g(x')}{(L - 2\,|\,x\,|)} \left(\mathrm{Ei}\left(\frac{2\pi i n (x - x')}{2\,|\,x\,|} \right) - \mathrm{Ei}\left(\frac{2\pi i n (x - x')}{L} \right) \right) \\
& + \int_{|x|}^{\frac{L}{2}} \mathrm{d}x' \frac{g(x')}{(L - 2\,|\,x\,|)} \left(\mathrm{Ei}\left(\frac{2\pi i n (x - x')}{2x'} \right) - \mathrm{Ei}\left(\frac{2\pi i n (x - x')}{L} \right) \right) \\
& + \int_{-\frac{L}{2}}^{-|x|} \mathrm{d}x' \frac{g(x')}{(L - 2\,|\,x\,|)} \left(\mathrm{Ei}\left(-\frac{2\pi i n (x - x')}{2x'} \right) - \mathrm{Ei}\left(\frac{2\pi i n (x - x')}{L} \right) \right)
\end{aligned}
\tag{8}
$$

This is an interesting conclusion, as it seems that the Fourier series has been transformed without changing the value of $g(x)$, and L is still a free parameter in the equation. It would seem natural to analyze the equation by taking the limit as $L \to \infty$, but this proves difficult, as one must account for how the ratio of $\frac{n}{L}$ evolves as one sums in n towards $\pm\infty$, as this is a ratio of two different variables both limiting towards ∞. As well, the integrals in x' have upper and lower bounds that involve L, meaning that the bounds also go to $\pm\infty$.

There are other questions that are pertinent. From previous investigations on this blog, the Fourier series prior to integrating in L has the following form:

$$
g(x) = \lim_{N \to \infty} \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}x'\, g(x') \left[\frac{\sin\left(2\pi N \frac{x - x'}{L}\right) \cot\left(\pi \frac{x - x'}{L}\right)}{L} \right]
\tag{9}
$$

The previous discussion outlined the idea that this function behaves like a Dirac Delta function, but is not a Dirac delta function in that it is not 0 everywhere and infinite at some value, but instead is an infinite frequency sine wave with a divergence at $x' = x$ such that when integrating, all values average to 0 (by the sine wave) but the $x' = x$ value which is infinite and does not average out. Averaging this function in L as we have done above should not change this behaviour, particularly as long as $N \gg L$. It is often stated that taking the L in a Fourier Series to infinity leads to the Fourier transform - this would be an interesting idea to explore in this alternative formulation, though it will not be discussed here.

It would be worthwhile to evaluate Equation 8 for a test function, such as $g(x) = x$. We evaluate the integrals and set $x \to \frac{\pi}{2}$ and $L \to 2\pi$. We then get an intriguing expression for π:

$$
\begin{aligned}
\pi &= \sum_{n \neq 0}^{(-\infty,\infty)} \frac{(-1)^n}{2n} \left(i^n (4i - 2n\pi) + (-1)^n (n\pi - i) + i n^2 \pi^2 \left(\mathrm{Ei}\left(\frac{i n \pi}{2} \right) - \mathrm{Ei}(i n \pi) \right) \right) \\
1 &= \sum_{n \neq 0}^{(-\infty,\infty)} \frac{(-1)^n}{2n\pi} \left(i^n (4i - 2n\pi) + (-1)^n (n\pi - i) + i n^2 \pi^2 \left(\mathrm{Ei}\left(\frac{i n \pi}{2} \right) - \mathrm{Ei}(i n \pi) \right) \right)
\end{aligned}
\tag{10}
$$

Many other expressions are possible for $g(x) = x$, and countless more are possible for different $g(x)$ functions. It may also be possible to integrate Equation 8 again in terms of L and apply the same formalism to obtain more alternative Fourier Series expressions. This concludes this blog post.