Title: Trigonometric Formulas in the Complex Plane
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I've recently been posed a problem that involves calculating a closed form solution for a sum whose terms are complex exponentials. For example, consider the general form of the following:

$$S_N\left(x\right) = \sum_{n=1}^{N} e^{inx} \tag{1}$$

where x is a real number. Now we expand e^{inx} in a Taylor Series and then interchanging the order of summation:

$$\begin{aligned} S_n\left(x\right) &= \sum_{s=0}^{\infty} \frac{\left(ix\right)^s}{s!} \sum_{n=1}^{N} n^s \\ &= \sum_{s=0}^{\infty} \frac{\left(ix\right)^s}{s!} \left(\frac{N^{s+1}-1}{s+1} + \frac{N^s+1}{2} + \sum_{k=1}^{\left\lfloor \frac{s}{2} \right\rfloor} \frac{B_{2k} s!}{(2k)!\left(s-2k+1\right)!} \left(N^{s-2k+1}-1\right) \right) \end{aligned} \tag{2}$$

where the second line follows from the solution to the power sum that I posted about previously (where in that post, we note that $B_{2n-1} = 0$ for $n \geq 2$. First two terms come in rather nicely, but the last (summation) term over the Bernoulli numbers presents a problem for the general solution. Now we zero in on ways to simplify the summation term and make a closed form solution to Equation 1 tenable. First we implement the integral expression of the Bernoulli Numbers taken from Wikipedia:

$$\begin{aligned} B_{2k} &= 4k\left(-1\right)^{k+1} \int_0^{\infty} \frac{t^{2k-1}}{e^{2\pi t}-1} \mathrm{d}t \\ &= \frac{4k\left(-1\right)^{k+1}}{\left(2\pi\right)^{2k}} \Gamma\left(2k\right) \zeta\left(2k\right) \\ &= \frac{\left(-1\right)^{k+1} 2\left(2k\right)!}{\left(2\pi\right)^{2k}} \zeta\left(2k\right) \end{aligned} \tag{3}$$

where the last line was derived by expanding the denominator in the first line as a geometric series and then performing the integration with the appropriate scaled variable. While the second line is a useful simplified form, summing with the Riemann Zeta function substituted into the second line of Equation 2 is unlikely to yield closed form results. We instead substitute the first line of

Equation 3 into the second line of Equation 2:

$$\begin{aligned}(iNx)^s \sum_{k=1}^{\lfloor \frac{s}{2} \rfloor} \frac{B_{2k}}{(2k)!\,(s-2k+1)!} N^{1-2k} &= \int_0^\infty \mathrm{d}t \frac{4\,(iNx)^s\,N}{t\,(e^{2\pi t}-1)} \sum_{k=1}^{\lfloor \frac{s}{2} \rfloor} \frac{k\,(-1)^{k+1}}{(2k)!\,(s-2k+1)!} \left(\frac{t}{N}\right)^{2k} \\ &= 2\frac{(iNx)^s}{s!} \int_0^\infty \mathrm{d}t \frac{\left(1+\frac{t^2}{N^2}\right)^{\frac{s}{2}} \sin\left(s \arctan\left(\frac{t}{N}\right)\right)}{(e^{2\pi t}-1)} \\ &= -i\frac{(iNx)^s}{s!} \int_0^\infty \mathrm{d}t \frac{\left(1+i\frac{t}{N}\right)^s - \left(1-i\frac{t}{N}\right)^s}{(e^{2\pi t}-1)}\end{aligned} \tag{4}$$

where the second line equality is taken from Mathematica. The third line equality manipulation is true by taking $\arctan(z) = \frac{i}{2}\ln\left(\frac{i+z}{i-z}\right)$ and $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$. The next equality is taken by interchanging the order of summation in s and integration in t:

$$\begin{aligned}S_N(x) &= \frac{e^{iNx}-e^{ix}}{ix} + \frac{1}{2}\left(e^{iNx}+e^{ix}\right) + 2i\left(e^{iNx}-e^{ix}\right)\int_0^\infty \mathrm{d}t \frac{\sinh(xt)}{(e^{2\pi t}-1)} \\ &= \frac{e^{iNx}-e^{ix}}{ix} + \frac{1}{2}\left(e^{iNx}+e^{ix}\right) + 2i\left(e^{iNx}-e^{ix}\right)\left(\frac{1}{2x} - \frac{1}{4}\cot\left(\frac{x}{2}\right)\right) \\ &= \frac{1}{2}\left(e^{iNx}+e^{ix}\right) - \frac{1}{2}i\left(e^{iNx}-e^{ix}\right)\cot\left(\frac{x}{2}\right)\end{aligned} \tag{5}$$

This solution is useful for a variety of problems - imagine the Fourier Series that we could model in closed form! This solution only works for $\mid x \mid \leq 2\pi$. As well, it is useful because it is implied that the real part of the solution is the sum of cosines while the imaginary part of the solution is the sum of sines. Finally, there is also some interesting investigation in the asymptotic case that $x = \frac{2\pi}{N}$ as $N \to \infty$. The real and imaginary parts of the solution imply that:

$$\begin{aligned}\sum_{n=1}^{N} \cos(nx) &= \frac{1}{2}\left(\cos(Nx)+\cos(x)\right) + \frac{1}{2}\left(\sin(Nx)-\sin(x)\right)\cot\left(\frac{x}{2}\right) \\ \sum_{n=1}^{N} \sin(nx) &= \frac{1}{2}\left(\sin(Nx)+\sin(x)\right) - \frac{1}{2}\left(\cos(Nx)-\cos(x)\right)\cot\left(\frac{x}{2}\right)\end{aligned} \tag{6}$$

Now for the question of the fate of the solution for $x = \frac{2\pi}{N}$ as $N \to \infty$, we give the following (where $\cot(x) = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} + \ldots$):

$$\begin{aligned}\lim_{N\to\infty} S_N &= 1 + \frac{1}{2}i\left(\frac{2\pi i}{N}\right)\left(\frac{N}{\pi}\right) \\ &= 0\end{aligned} \tag{7}$$

As expected, for perfect geometry, where every complex exponential in one direction has another complex exponential in the opposite direction, the solution has symmetry. In the less restrictive case, where $N \to \infty$ but $x \neq \frac{2\pi}{N}$, we get the case of a Fourier Series with coefficients of $c_n = 1$, such that:

$$\lim_{N\to\infty} \sum_{n=1}^{N} e^{inx} = \lim_{N\to\infty}\left(\frac{1}{2}\left(e^{iNx}+e^{ix}\right) - \frac{1}{2}i\left(e^{iNx}-e^{ix}\right)\cot\left(\frac{x}{2}\right)\right) \tag{8}$$

With the exact solution coming for integer N, yielding the use of calculus limit notation slightly inaccurate. However, these complex exponential trigonometric findings are interesting considering the geometry of the complex plane - I plan to make future posts elaborating on this. This concludes this blog post.