Title: Examining Fractional Calculus

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Recall the fractional integral formula:

$$(J^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, \mathrm{d}t \tag{1}$$

The issue I will examine for this post is: what is the derivative of the fractional integral with respect to the integration order α? First, we examine:

$$\begin{aligned} \partial_\alpha (J^{\alpha} f)(x) &= -\frac{\Gamma'(\alpha)}{(\Gamma(\alpha))^2} \int_0^x (x-t)^{\alpha-1} f(t)\, \mathrm{d}t + \frac{1}{\Gamma(\alpha)} \int_0^x \ln(x-t)(x-t)^{\alpha-1} f(t)\, \mathrm{d}t \\ &= \left(\ln(x) - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}\right)(J^{\alpha} f)(x) + \frac{x^{\alpha}}{\Gamma(\alpha)} \int_0^1 \mathrm{d}u\, \ln(1-u)(1-u)^{\alpha-1} f(xu) \end{aligned} \tag{2}$$

Now we can specify this general formula for the derivative with respect to the integration order α - let $\alpha = 1$ and $f(t) = \sum_{p=0}^{\infty} a_p x^p$ is analytic. Then, since $\Gamma(1) = 1$ and $\Gamma'(1) = -\gamma$, where γ is the Euler Mascheroni constant. Then:

$$\begin{aligned} (\partial_\alpha (J^{\alpha} f_{\text{analytic}})(x))_{\alpha=1} &= \sum_{p=0}^{\infty} \frac{(\ln(x)+\gamma)}{p+1} a_p x^{p+1} + \sum_{p=0}^{\infty} a_p x^{p+1} \int_0^1 \mathrm{d}u\, u^p \ln(1-u) \\ &= \sum_{p=0}^{\infty} \frac{(\ln(x)+\gamma)}{p+1} a_p x^{p+1} - \sum_{p=0}^{\infty} a_p x^{p+1} \sum_{r=1}^{\infty} \frac{\int_0^1 \mathrm{d}u\, u^{p+r}}{r} \\ &= \sum_{p=0}^{\infty} \frac{(\ln(x)+\gamma)}{p+1} a_p x^{p+1} - \sum_{p=0}^{\infty} \sum_{r=1}^{\infty} \frac{a_p x^{p+1}}{r(p+r+1)} \\ &= \sum_{p=0}^{\infty} \frac{(\ln(x)+\gamma)}{p+1} a_p x^{p+1} - \sum_{p=0}^{\infty} \frac{a_p x^{p+1}}{p+1} \sum_{r=1}^{\infty} \left(\frac{1}{r} - \frac{1}{r+p+1}\right) \\ &= (\ln(x)+\gamma) \sum_{p=0}^{\infty} \frac{a_p x^{p+1}}{p+1} - \sum_{p=0}^{\infty} \frac{a_p x^{p+1}}{p+1} H_{p+1} \end{aligned} \tag{3}$$

where $H_{p+1} = \sum_{k=1}^{p+1} \frac{1}{k}$ is the $(p+1)$th Harmonic Number. We have not yet specified a_p, which defines the analytic function $f(t)$. Let $a_p = \frac{(-1)^p}{p!}$ such that $f(t) = e^{-t}$. Let the left side (derivative with respect to α) be 0 for some right side $x = x_0$, defining the position for which there is no change as α changes at $\alpha = 1$. Now we evaluate:

$$\begin{aligned} 0 &= -(\ln(x_0)+\gamma) \sum_{p=0}^{\infty} \frac{(-x_0)^{p+1}}{(p+1)!} + \sum_{p=0}^{\infty} \frac{(-x_0)^{p+1}}{(p+1)!} H_{p+1} \\ &= -(\ln(x_0)+\gamma)\left(e^{-x_0} - 1\right) + e^{-x_0}(\gamma + \ln(x_0) - I_{\cosh}(x_0) - I_{\sinh}(x_0)) \end{aligned} \tag{4}$$

where the second term in the second line was attained by computer algebra software (I have also confirmed numerically). Then, we get the transcendental equation for x_0:

$$\begin{aligned}\ln(x_0) + \gamma &= e^{-x_0}\left(I_{\text{cosh}}(x_0) + I_{\text{sinh}}(x_0)\right) \\ f_{\text{log}}(x_0) &= f_{\text{trigh}}(x_0)\end{aligned} \tag{5}$$

where:

$$\begin{aligned}I_{\text{cosh}}(x) &= \gamma + \ln(x) + \int_0^x \frac{\cosh(t) - 1}{t}\mathrm{d}t \\ I_{\text{sinh}}(x) &= \int_0^x \frac{\sinh(t)}{t}\mathrm{d}t\end{aligned} \tag{6}$$

There exists a trivial solution where $x_0 = 0$. The non-trivial solution (for $x_0 > 0$) is $x_0 \approx 1.167$. The expectation based on Equation 5 is that there is only one non-trivial solution, since $\partial_x f_{\text{log}}(x) = \partial_x \ln(x) = \frac{1}{x} > 0 \,\forall\, x \in \mathbb{R}_+$ and while:

$$\lim_{x\to 0} \partial_x f_{\text{trigh}}(x) = \lim_{x\to 0}\left(-f_{\text{trigh}}(x) + \partial_x f_{\text{log}}(x)\right) > \partial_x f_{\text{log}}(x) > 0 \tag{7}$$

and:

$$\begin{aligned}&\lim_{x\to\infty} f_{\text{trigh}}(x) = 0 \\ &\lim_{x\to\infty} f_{\text{log}}(x) \to \infty\end{aligned} \tag{8}$$

such that $f_{\text{log}}(0) = f_{\text{trigh}}(0)$, trivially, and f_{trigh} rises faster, but ends at 0 while f_{log} ends at ∞, and thus it is likely that they intersect at a single non-trivial point (and indeed, with the decaying exponential, the derivative of $f_{\text{trigh}}(x)$ is purely negative after some threshold $x \approx 1 < x_0$). We can apply Equation 3 to many other definitions of a_p and $f(t)$. However, I would like to finish by finding the function $f(t)$ for which *every* x in $(J^\alpha f)(x)$ has no change with α at $\alpha = 1$ in Equation 2. To do this, we need to find the function $f(t)$ such that the following is true for all x:

$$\gamma \int_0^x \mathrm{d}t\, f(t) + \int_0^x \mathrm{d}t\, \ln(x - t)\, f(t) = 0 \tag{9}$$

This integral equation requires some ingenuity to equivalently represent as a differential equation - this will be the focus of the next post. To start with, if we take the derivative in x to the left side, we get:

$$\begin{aligned}0 &= \gamma f(x) + \partial_x\left(\ln(x)\int_0^x \mathrm{d}t\, f(t) + x\int_0^1 \mathrm{d}u \ln(1-u)\, f(xu)\right) \\ &= \gamma f(x) + \ln(x)\, f(x) + \frac{1}{x}\int_0^x \mathrm{d}t\, f(t) + \int_0^1 \mathrm{d}u \ln(1-u)\, f'(xu)(1+xu) \\ \implies 0 &= \partial_x\left(\gamma x f(x) + x\ln(x)\, f(x)\right) + f(x) + \partial_x\left(x\int_0^1 \mathrm{d}u \ln(1-u)\, f'(xu)(1+xu)\right) \\ &= \partial_x\left(\gamma x f(x) + x\ln(x)\, f(x)\right) + f(x) + \int_0^1 \mathrm{d}u \ln(1-u)\, g_1'(xu)(1+xu)\end{aligned} \tag{10}$$

where the $g_1(xu) = f'(xu)(1+xu)$. Now that there is only one integral left, we have to find ways to isolate for the integrand. This will be the topic of my next post. This concludes this blog post.