Title: Fractional Calculus and the Stirling Numbers
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In a previous post, I verified that for $f(t) = \sum_{n=0}^{\infty} a_n t^n$, the fractional integral is:

$$(J^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \mathrm{d}t\ (x-t)^{\alpha-1} f(t) = \frac{x^\alpha}{\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{a_m\, m! x^m}{(\alpha)_{m+1}} \tag{1}$$

As seen in my last post, there is a way to evaluate logarithmic integrals from this formulation. We multiply both sides by $\frac{\Gamma(\alpha)}{x^\alpha}$, then differentiate with respect to α p times, and then set $\alpha = 1$. These actions lead to the following relationship:

$$\partial_\alpha^p \left(\int_0^1 \mathrm{d}t\ (1-t)^{\alpha-1}\ f(xt) \right)_{\alpha\to 1} = \partial_\alpha^p \left(\sum_{m=0}^{\infty} \frac{a_m\, m! x^m}{(\alpha)_{m+1}} \right)_{\alpha\to 1} \tag{2}$$

Now to evaluate the left side, where $f(xt) = e^{xt} - \sum_{m=0}^{p-1} \frac{(xt)^m}{m!} = \sum_{m=p}^{\infty} \frac{(xt)^m}{m!}$:

$$\begin{aligned}
\partial_\alpha^p \left(\int_0^1 \mathrm{d}t\ (1-t)^{\alpha-1}\ f(xt) \right)_{\alpha\to 1} &= \partial_\alpha^p \left(\sum_{n=0}^{\infty} \frac{(1-\alpha)_n}{n!} \int_0^1 \mathrm{d}t\, t^n\, f(xt) \right)_{\alpha\to 1} \\
\implies \int_0^1 \mathrm{d}t\ \ln^p(1-t)\, f(xt) &= \partial_\alpha^p \left(\sum_{n=0}^{\infty} \frac{(1-\alpha)_n}{n!} \sum_{m=0}^{\infty} \left(\frac{x^{m+p}}{(m+p)!} \frac{1}{m+p+n+1} \right) \right)_{\alpha\to 1} \\
&= \partial_\alpha^p \left(\sum_{n=0}^{\infty} \frac{(1-\alpha)_n}{n!} \sum_{m=0}^{\infty} \left(\frac{x^{m+p}}{(m+p)!} \frac{1}{m+p+n+1} \right) \right)_{\alpha\to 1}
\end{aligned} \tag{3}$$

where $\gamma(n+1, w) = \int_0^w \mathrm{d}t\, t^n e^{-t}$ is the upper incomplete gamma function. Now we calculate from the left side again, using the identity that $\frac{(-\ln(1-t))^p}{p!} = \sum_{m=p}^{\infty} \frac{\left[{m \atop p}\right]}{m!} t^m$:

$$\begin{aligned}
\int_0^1 \mathrm{d}t\ \ln^p(1-t)\, f(xt) &= (-1)^p \sum_{n=p}^{\infty} \frac{p!}{n!} \left[{n \atop p}\right] \int_0^1 \mathrm{d}t\, t^n f(xt) \\
&= (-1)^p \sum_{n=p}^{\infty} \frac{p!}{n!} \left[{n \atop p}\right] \sum_{m=0}^{\infty} \frac{x^{m+p}}{(m+p)!} \frac{1}{m+p+n+1} \\
&= \sum_{m=0}^{\infty} \sum_{n=p}^{\infty} (-1)^p \frac{p!}{n!} \left[{n \atop p}\right] \frac{x^{m+p}}{(m+p)!} \frac{1}{m+p+n+1}
\end{aligned} \tag{4}$$

Now we evaluate the right hand side of Equation 2 for the chosen $f(t)$:

$$\int_0^1 \mathrm{d}t\, \ln^p(1-t)\, f(xt) = \sum_{m=0}^{\infty}\left(\partial_\alpha^p \frac{x^{m+p}}{(\alpha)_{m+p+1}}\right)_{\alpha\to 1} \tag{5}$$

Equating powers between Equation 3 and Equation 5, we get:

$$\left(\partial_\alpha^p \frac{1}{(\alpha)_{m+p+1}}\right)_{\alpha\to 1} = \frac{(-1)^p\, p!}{(m+p)!}\sum_{n=p}^{\infty}\frac{\left[{n \atop p}\right]}{n!}\frac{1}{n+m+p+1} = \frac{1}{(m+p)!}\sum_{n=0}^{\infty}\frac{\left(\partial_\alpha^p (1-\alpha)_n\right)_{\alpha\to 1}}{n!\,(n+m+p+1)} \tag{6}$$

where $\left[{n \atop p}\right]$ are the unsigned Stirling Numbers of the First Kind. These are quite unexpected relations between derivatives of Pochhammer Symbols and the Stirling Numbers. The last identity is the most surprising, as a naive assessment would suggest that $\lim_{\alpha\to 1}\partial_\alpha^p (1-\alpha)_n = -(1-\alpha)_n\left(\psi_0(1-\alpha+n) - \psi\right.$ 0, as the Pochhammer Symbol persists for all derivatives and goes to 0 in the limit. However, because of the Polygamma function, the $\lim_{\alpha\to 1}\psi_0(1-\alpha)\to\infty$, the limits cancel in the following manner, as an example (let $p=1$:

$$\lim_{\alpha\to 1}\frac{\partial_\alpha (1-\alpha)_n}{\Gamma(1-\alpha+n)} = \lim_{\alpha\to 1}\frac{\left(-(1-\alpha)_n\left(\psi_0(1-\alpha+n)-\psi_0(1-\alpha)\right)\right)}{\Gamma(1-\alpha+n)} = \lim_{\alpha\to 1}\frac{\Gamma'(1-\alpha)}{\Gamma^2(1-\alpha)} = -1$$

where the last identity arises due to $\ln(\Gamma(x)) = \ln(\Gamma(x+1)) - \ln(x)$, such that $\partial_x \ln(\Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)} = \frac{\Gamma'(x+1)}{\Gamma(x+1)} - -\frac{1}{x}$. Then:

$$\begin{aligned}\lim_{x\to 0}\left(\frac{\Gamma'(x)}{\Gamma^2(x)}\right) &= \lim_{x\to 0}\left(\frac{\Gamma'(x)}{\Gamma(x)}\right)\frac{x}{\Gamma(x+1)}\\ &= \lim_{x\to 0}\left(\frac{\Gamma'(x+1)}{\Gamma(x+1)} - \frac{1}{x}\right)\frac{x}{\Gamma(x+1)}\\ &= -1\end{aligned} \tag{7}$$

Now, the right side of Equation 6 has these kinds of limits being evaluated for every term, making these expressions quite profound. For $p=1$, we have:

$$-\frac{\left(\psi_0(\alpha+m+2) - \psi_0(\alpha)\right)}{(m+2)!} = -\frac{1}{(m+1)!}\sum_{n=1}^{\infty}\left(\frac{1}{n(n+m+2)}\right) \tag{8}$$

where we note that for the far right side of Equation 6, the 0th order term vanishes, as

$$\partial_\alpha (1-\alpha)_0 = 0$$

so that the middle and right side expressions in Equation 6 come to the right side expression in Equation 8. We will find that the middle and the right side of Equation 6 becomes similar again will for $p=2$:

$$\begin{aligned}\frac{2}{(m+2)!}\sum_{n=2}^{\infty}\left(\frac{H_{n-1}}{n(n+m+3)}\right) &= \frac{2}{(m+2)!}\sum_{n=1}^{\infty}\left(\frac{\gamma+\psi_0(n)}{n(n+m+3)}\right)\\ \implies \frac{2}{(m+2)!}\sum_{n=2}^{\infty}\left(\frac{H_{n-1} - (\gamma+\psi_0(n))}{n(n+m+3)}\right) &= \frac{\gamma+\psi_0(1)}{m+4} = 0\end{aligned} \tag{9}$$

For $p = 3$:

$$\frac{-6}{(m+3)!}\sum_{n=3}^{\infty}\left(\frac{1}{2}\frac{H_{n-1}^2 - H_{n-1}^{(2)}}{n\,(n+m+4)}\right) = \frac{-6}{(m+3)!}\sum_{n=1}^{\infty}\left(\frac{1}{2}\frac{\gamma - \pi^2 + \psi_0\,(n)\,(2\gamma + \psi_0\,(n) + \psi_1\,(n))}{n\,(n+m+4)}\right)$$

$$\frac{-6}{(m+3)!}\sum_{n=3}^{\infty}\frac{1}{2}\frac{H_{n-1}^2 - H_{n-1}^{(2)} - \gamma + \pi^2 - \psi_0\,(n)\,(\gamma + \psi_0\,(n)) - \psi_1\,(n)}{n\,(n+m+4)}$$

$$= \frac{-6}{(m+3)!}\sum_{n=1}^{2}\left(\frac{1}{2}\frac{\gamma - \pi^2 + \psi_0\,(n)\,(2\gamma + \psi_0\,(n) + \psi_1\,(n))}{n\,(n+m+4)}\right)$$

$$= 0 \tag{10}$$

The observation that the first $p-1$ terms of the far right side of Equation 6 keep going to zero suggests that possibly we may consider:

$$\begin{bmatrix} n \\ p \end{bmatrix} = \lim_{\alpha \to 1}\left(\frac{\partial_\alpha^p\,(1-\alpha)_n}{(-1)^p\,p!}\right)$$

This equation seems to hold for all n, p combinations that I have tried. Further, it seems rather convenient to have a straightforward, general way to compute the Stirling Number expression in terms of Polygamma's. This solution brings the Stirling Numbers of the First Kind into the domain of calculus, a highly desirable result. There are now many possibilities to explore with different choices for $f\,(t)$.