Title: Integrating the Polylogarithmic Function 1
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In the last post, I alluded to the idea that integrating the polylogarithm function is somewhat more challenging than integrating the simple logarithm. In this post, I address this point and provide strategies for completing polylogarithm integrals via rather different (less brute-force, computational based) strategy than my recommended algorithm (see previous post) for computing simpler logarithmic integrals. First, acknowledge that polylogarithm is defined as follows, with the quite elegant derivative relationship:

$$\begin{aligned} \mathrm{Li}_s(x) &= \sum_{k=1}^{\infty} \frac{x^k}{k^s} \\ \implies \partial_x \mathrm{Li}_s(x) &= \frac{\mathrm{Li}_{s-1}(x)}{x} \end{aligned} \tag{1}$$

With this function definition given, it is clear that $\mathrm{Li}_1 = -\ln(1-x)$ and that $\mathrm{Li}_s(0) = 0$ for any s. Also, the derivative relation creates some useful algebra when integrating by parts. For example:

$$\begin{aligned} \int_0^x \mathrm{d}t\, \mathrm{Li}_2(t) &= \left[t\mathrm{Li}_2(t)\right]_{t\to 0}^{t\to x} - \int_0^x \mathrm{d}t\, \mathrm{Li}_1(t) \\ &= x\mathrm{Li}_2(x) + \int_0^x \mathrm{d}t\, \ln(1-t) \\ &= x\mathrm{Li}_2(x) - (1-x)\ln(1-x) - x \end{aligned} \tag{2}$$

This "polylogarithm integration by parts" algebra is the main method towards a solution for polylogarithm integrals that I will demonstrate in this blog post. Simply put, integrate by parts until one attains a $\mathrm{Li}_1(x)$ that can be converted to a logarithm. Then one can focus on the remaining polylogarithm term. Now that we have introduced this method, I will now tackle a relatively more challenging problem - consider the following integral:

$$\begin{aligned} I_1(x) &= \int_0^x \mathrm{d}t\, \mathrm{Li}_2(t)\,\mathrm{Li}_2(1-t) \\ &= \left[\left(t\mathrm{Li}_2(t) - (1-t)\ln(1-t) - t\right)\mathrm{Li}_2(1-t)\right]_{t\to 0}^{t\to x} \\ &\quad - \int_0^x \mathrm{d}t\, \left(\frac{t}{1-t}\mathrm{Li}_2(t)\ln(t) - \ln(1-t)\ln(t) - \frac{t}{1-t}\ln(t)\right) \\ &= x\mathrm{Li}_2(x)\,\mathrm{Li}_2(1-x) - (1-x)\ln(1-x)\,\mathrm{Li}_2(1-x) - x\mathrm{Li}_2(1-x) \\ &\quad - \int_0^x \mathrm{d}t\, \left(\frac{t}{1-t}\mathrm{Li}_2(t)\ln(t) - \ln(1-t)\ln(t) - \frac{t}{1-t}\ln(t)\right) \end{aligned} \tag{3}$$

Expand $\frac{1}{1-t} = \sum_{k=0}^{\infty} t^k$ as the geometric series. Then, the following integral identities are useful:

$$\begin{aligned}\int \mathrm{d}t\, t^{1+k} \mathrm{Li}_2(t) &= \frac{\mathrm{B}_t(3+k,0) + t^{2+k}\left(\ln(1-t) + (2+k)\,\mathrm{Li}_2(t)\right)}{(2+k)^2} + C \\ \int_0^x \mathrm{d}t\, \ln(t)\ln(1-t) &= -\mathrm{Li}_2(x) - (1-x)\ln(1-x)(\ln(x)-1) - x(\ln(x)-2) \\ \int_0^x \mathrm{d}t\, \frac{t}{1-t}\ln(t) &= -\ln(1-x)\ln(x) - x\ln(x) - \mathrm{Li}_2(x) + x\end{aligned} \tag{4}$$

where the first line of Equation 4 can be attained by integrating by parts twice and differentiating the polylogarithm twice, noting that $\mathrm{Li}_0(t) = \frac{t}{1-t}$ (where $\mathrm{B}_t(3+k,0)$ is the incomplete beta function). The second line result is purely logarithmic and can be readily attained using the methods of the recent previous posts, and the third line can be attained through carrying out the geometric series expansion, doing integration by parts, then doing the integral and evaluating the series expansion. With these identities stated, we can now evaluate Equation 3 in terms of a series expansion:

$$\begin{aligned}I_1(x) = {} & x\mathrm{Li}_2(x)\,\mathrm{Li}_2(1-x) - (1-x)\ln(1-x)\,\mathrm{Li}_2(1-x) - x\mathrm{Li}_2(1-x) \\ & - \mathrm{Li}_2(x) - (1-x)\ln(1-x)(\ln(x)-1) - x(\ln(x)-2) \\ & - \ln(1-x)\ln(x) - x\ln(x) - \mathrm{Li}_2(x) + x \\ & - \sum_{k=0}^{\infty} \frac{\mathrm{B}_x(3+k,0)\ln(x) + x^{2+k}\ln(x)\left(\ln(1-x) + (2+k)\,\mathrm{Li}_2(x)\right)}{(2+k)^2} \\ & + \sum_{k=0}^{\infty} \int_0^x \mathrm{d}t\, \frac{\frac{\mathrm{B}_t(3+k,0)}{t} + t^{1+k}\left(\ln(1-t) + (2+k)\,\mathrm{Li}_2(t)\right)}{(2+k)^2}\end{aligned} \tag{5}$$

Now the solution is on the run. The critical step is to find that $\int_0^x \mathrm{d}t\, \frac{\mathrm{B}_t(3+k,0)}{t} = x^{3+k}\Phi(x,2,3+k)$, where $\Phi(x,s,\alpha) = \sum_{n=0}^{\infty} \frac{x^n}{(n+\alpha)^s}$ is the Hurwitz Lerch Phi (Lerch Transcendent) function (try series expanding the incomplete beta function, then integrate and the Lerch Transcendent will arise). An additional integral is $\int_0^x \mathrm{d}t\, t^{1+k}\ln(1-t) = x^{2+k}\frac{(x\Phi(x,1,3+k)+\ln(1-x))}{2+k}$ (again, series expand the logarithm and integrate). Now, we present the full solution, numerically checked.

$$\begin{aligned}\int_0^x \mathrm{d}t\, \mathrm{Li}_2(t)\,\mathrm{Li}_2(1-t) = {} & x\mathrm{Li}_2(x)\,\mathrm{Li}_2(1-x) - (1-x)\ln(1-x)\,\mathrm{Li}_2(1-x) - x\mathrm{Li}_2(1-x) \\ & - \mathrm{Li}_2(x) - (1-x)\ln(1-x)(\ln(x)-1) - x(\ln(x)-2) \\ & - \ln(1-x)\ln(x) - x\ln(x) - \mathrm{Li}_2(x) + x \\ & - \sum_{k=0}^{\infty} \frac{\mathrm{B}_x(3+k,0)\ln(x) + x^{2+k}\ln(x)\left(\ln(1-x) + (2+k)\,\mathrm{Li}_2(x)\right)}{(2+k)^2} \\ & + \sum_{k=0}^{\infty} \frac{x^{3+k}\Phi(x,2,3+k)}{(2+k)^2} + \frac{x^{2+k}\left(x\Phi(x,1,3+k) + \ln(1-x)\right)}{(2+k)^3} \\ & + \sum_{k=0}^{\infty} \frac{\mathrm{B}_x(3+k,0)}{(2+k)^3} + \frac{x^{2+k}\left(\ln(1-x) + (2+k)\,\mathrm{Li}_2(x)\right)}{(2+k)^3}\end{aligned} \tag{6}$$

Even a relatively simple polylogarithmic integral like $I_1(x)$ provides substantial complexity relative to the relatively compact form of the analogous logarithmic integral. This concludes this blog post.