Title: Revisiting Fractional Integration
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July 1, 2025

Recall fractional integration formula:

$$\left(J^{(\alpha)}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \mathrm{d}t\,(x-t)^{\alpha-1} f(t) \tag{1}$$

The first question I set out to answer is what the formula actually means in the context of $\alpha \in \mathbb{Z}$, $\alpha \geq 2$. I was able to answer this question quickly through attempting $\alpha = 2$ for $f(x) = e^x$. In this case:

$$\left(J^2 e^x\right) = \int_0^x \mathrm{d}x' \int_0^{x'} \mathrm{d}t\, e^t = e^x - x - 1 = \int_0^x \mathrm{d}t\,(x-t)\,e^t \tag{2}$$

An insight that can be drawn from this example is the idea that because $J^{\delta+\beta} = J^{\delta}J^{\beta}$, we can carry out the kind of repeated integration characteristic of the first equality in Equation 2 for cases where $\alpha \in \mathbb{Z}, \alpha \geq 2$ such that $\beta = \lfloor \alpha \rfloor$, and $\delta = \alpha - \lfloor \alpha \rfloor < 1$. Then, all that is left is to interpret J^{δ}, which is governed simply by Equation 1.

Now that this is clear, I will investigate some results from this fractional integration formula. First:

$$\begin{aligned}\sum_{n=1}^{N} \left(J^n f\right)(x) &= \int_0^x \mathrm{d}t f(t) \sum_{n=1}^{N} \frac{(x-t)^{n-1}}{(n-1)!} \\ &= \int_0^x \mathrm{d}t \frac{\Gamma(N, x-t)}{\Gamma(N)} e^{x-t} f(t) \\ &= \frac{e^x}{\Gamma(N)} \int_0^x \mathrm{d}t\, \Gamma(N, x-t)\, e^{-t} f(t)\end{aligned} \tag{3}$$

where $\Gamma(N, x-t) = \int_x^{\infty} t^{N-1} e^{-t}$ is the incomplete gamma function. Now we can be creative with this formula - let $f(t) = e^t$. First note that $J^n\, e^x = e^x - \sum_{k=0}^{n-1} \frac{x^k}{k!}$. Then, we find:

$$\begin{aligned}
\int_0^x \mathrm{d}t\, \Gamma(N, x-t) &= (N-1)!\, e^{-x} \sum_{n=1}^{N} (J^n f)(x) \\
&= (N-1)!\, e^{-x} \sum_{n=1}^{N} \left(e^x - \sum_{k=0}^{n-1} \frac{x^k}{k!} \right) \\
&= N! - (N-1)!\, e^{-x} \sum_{k=0}^{N} \frac{x^k}{k!} \sum_{n=k+1}^{N} (1) \\
&= N! - (N-1)!\, e^{-x} \sum_{k=0}^{N} \frac{x^k}{k!} (N-k) \\
&= N! \left(1 - \, e^{-x} \sum_{k=0}^{N} \frac{x^k}{k!} + e^{-x} \sum_{k_1=1}^{N} \frac{x^{k_1}}{(k_1-1)!} \right) \\
&= N! \left(1 - \frac{\Gamma(N+1, x)}{\Gamma(N+1)} + e^{-x} \frac{x}{N} \sum_{k=1}^{N} \left(\frac{x^{k-1}}{(k-1)!} \right) \right) \\
&= N! - \Gamma(N+1, x) + x\Gamma(N, x)
\end{aligned} \tag{4}$$

I've confirmed this result with Mathematica. Another (simpler) calculation is possible for $f(x) = x^p$, where p is a positive integer (since $J^n t^p = \frac{p!}{(n+p)!} t^{n+p}$):

$$\begin{aligned}
\sum_{n=1}^{N} (J^n f)(x) &= \frac{e^x}{\Gamma(N)} \int_0^x \mathrm{d}t\, \Gamma(N, x-t)\, e^{-t}\, t^p \\
\implies \int_0^x \mathrm{d}t\, \Gamma(N, x-t)\, e^{-t}\, t^p &= p!\ (N-1)!\, e^{-x} \sum_{n=1}^{N} \frac{t^{n+p}}{(n+p)!} \\
&= p!\ (N-1)!\, e^{-x} \sum_{n=p+1}^{N+p} \frac{t^n}{n!} \\
&= p!\ (N-1)!\, e^{-x} \left(\sum_{n=0}^{N+p} \frac{t^n}{n!} - \sum_{n=0}^{p} \frac{t^n}{n!} \right) \\
&= p!\ (N-1)!\, e^{-x} \left(e^x \frac{\Gamma(N+p+1, x)}{\Gamma(N+p+1)} - e^x \frac{\Gamma(p+1, x)}{\Gamma(p+1)} \right) \\
&= \frac{\Gamma(N+p+1, x)}{N \binom{N+p}{p}} - (N-1)!\, \Gamma(p+1, x)
\end{aligned} \tag{5}$$

Which is a tricky result to derive otherwise. Another interesting result is if we make $f(t) = e^{ix}$ -

$(J^n e^{ix} = \frac{1}{i^n} e^{ix} - \sum_{k=0}^{n-1} \frac{x^k}{k!} i^{n-k})$:

$$\sum_{n=1}^{N} (J^n f)(x) = \frac{e^x}{\Gamma(N)} \int_0^x \mathrm{d}t\, \Gamma(N, x-t)\, e^{-t(1-i)}$$

$$\begin{aligned}
\implies \int_0^x \mathrm{d}t\, \Gamma(N, x-t)\, e^{-t(1-i)} &= (N-1)!\, e^{-x} \sum_{n=1}^{N} \left(\frac{e^{ix}}{i^n} - \sum_{k=0}^{n-1} \frac{x^k}{k!} (-i)^{n-k} \right) \\
&= (N-1)!\, e^{-x(1-i)} \sum_{n=1}^{N} (-i)^n \\
&\quad + (N-1)!\, e^{-x} \sum_{k=0}^{N} \frac{(ix)^k}{k!} \sum_{n=k+1}^{N} (-i)^n \\
&= (N-1)!\, e^{-x(1-i)} \sum_{n=1}^{N} (-i)^n \\
&\quad - (N-1)!\, e^{-x} \sum_{k=0}^{N} \frac{(ix)^k}{k!} \left(\frac{(-i)^{k+1} - (-i)^{N+1}}{1+i} \right) \\
&= -i\,(N-1)!\, e^{-x(1-i)} \frac{1-(-i)^N}{1+i} \\
&\quad + \frac{i}{1+i} (N-1)!\, e^{-x} \sum_{k=0}^{N} \frac{(x)^k}{k!} \\
&\quad + \frac{(-i)^{N+1}}{1+i} (N-1)!\, e^{-x} \sum_{k=0}^{N} \frac{(ix)^k}{k!} \\
&= (N-1)!\, e^{-x(1-i)} \frac{\left((-i) - (-i)^{N+1}\right)}{1+i} \\
&\quad + \frac{i}{1+i} \frac{\Gamma(N+1, x)}{N} \\
&\quad + \frac{(-i)^{N+1}}{1+i} e^{-x(1-i)} \frac{\Gamma(N+1, ix)}{N}
\end{aligned} \tag{6}$$

Well that was a marathon solution! The real part of this solution is $\int_0^x \mathrm{d}t\, \Gamma(N, x-t)\, e^{-t} \cos(t)$ and the imaginary part is $\int_0^x \mathrm{d}t\, \Gamma(N, x-t)\, e^{-t} \sin(t)$. Through repeated calculus methods, we have obtained the solutions of this complicated integral. In the next post I will cover more on the fractional calculus aspects of Equation 1. This concludes this blog post.