Title: Fractional Calculus Integrals

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In this post, I explore some of the philosophy surrounding fractional calculus and associated integrals. We recall the fractional integral formula of order α:

$$\left(J^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \mathrm{d}t \, (x-t)^{\alpha-1} f(t) \tag{1}$$

It has been proven elsewhere on this website that the following property (intuitively) holds true:

$$\left(J^{\beta}\left(J^{\alpha} f\right)\right)(x) = \left(J^{\alpha+\beta} f\right)(x) \tag{2}$$

With this said, we consider applying Equation 1 in the case that $0 < \alpha \leq 1$. Let's re-arrange, then implement the fractional power binomial series expansion:

$$\begin{aligned} \left(J^{\alpha} f\right)(x) &= \frac{x^{\alpha-1}}{\Gamma(\alpha)} \int_0^x \mathrm{d}t \, \left(1 - \frac{t}{x}\right)^{\alpha-1} f(t) \\ &= \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{(1-\alpha)_n}{n!} x^{\alpha-n-1} \int_0^x \mathrm{d}t \, t^n \, f(t) \end{aligned} \tag{3}$$

where $(1-\alpha)_n = \frac{\Gamma(1-\alpha+n)}{\Gamma(1-\alpha)}$ is the rising Pochhammer Symbol. It is interesting that the computational path for fractional calculus can still lead to (an infinite series of) integer power integrals. This integral form is common in analysis of probability density functions, where statistical moments are computed as integrals of integer power weightings of the probability density function over the variable domain. Now we assume that $f(x) = \sum_{m=0}^{\infty} a_m x^m$ is analytic and can be expressed as a power series. Then, we proceed:

$$\begin{aligned} \left(J^{\alpha} f\right)(x) &= \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{(1-\alpha)_n}{n!} x^{\alpha-n-1} \sum_{m=0}^{\infty} \frac{a_m x^{n+m+1}}{n+m+1} \\ &= \frac{x^{\alpha}}{\Gamma(\alpha)} \sum_{m=0}^{\infty} a_m x^m \sum_{n=0}^{\infty} \frac{(1-\alpha)_n}{n! \, (n+m+1)} \end{aligned} \tag{4}$$

Now we consider the special case of rational values of α with numerator 1 - that is, $\alpha = \frac{1}{k}$ for k as a natural number. We proceed further:

$$
\begin{aligned}
\left(J^{\alpha}f\right)(x) &= \frac{x^{\frac{1}{k}}}{\Gamma\left(\frac{1}{k}\right)}\sum_{m=0}^{\infty}a_m x^m\sum_{n=0}^{\infty}\frac{\left(\frac{k-1}{k}\right)_n}{n!\,(n+m+1)} \\
&= \frac{x^{\frac{1}{k}}}{\Gamma\left(\frac{1}{k}\right)}\sum_{m=0}^{\infty}a_m\left(\frac{\Gamma\left(\frac{1}{k}\right)m!}{\Gamma\left(m+1+\frac{1}{k}\right)}\right)x^m \\
&= x^{\frac{1}{k}}\sum_{m=0}^{\infty}\frac{a_m m!\,x^m}{\left(m+\frac{1}{k}\right)\Gamma\left(m+\frac{1}{k}\right)}
\end{aligned}
\tag{5}
$$

where the second line of Equation 5 was a solution that acquired from Mathematica. It turns out that there is no general formula for all k for $\Gamma\left(\frac{1}{k}\right)$, which is all that is needed to know $\Gamma\left(m+\frac{1}{k}\right)$, since in general $\Gamma(x+1)=x\Gamma(x)$ (i.e. $\Gamma\left(m+\frac{1}{k}\right)$ can be indexed down m times until we have only $\Gamma\left(\frac{1}{k}\right)$). However, there is a single special case that is known for any m - when $k=2$:

$$
\Gamma\left(\frac{r}{2}\right)=\frac{(r-2)!!}{2^{\frac{r-1}{2}}}\sqrt{\pi}
\tag{6}
$$

where $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$. We proceed with the special case of $k=2$, where $r\to 2m+1$:

$$
\begin{aligned}
\left(J^{\alpha}f\right)(x) &= \frac{x^{\frac{1}{2}}}{\sqrt{\pi}}\sum_{m=0}^{\infty}\frac{a_m\,(2m)!!\,x^m}{\left(m+\frac{1}{2}\right)(2m-1)!!} \\
&= \frac{2x^{\frac{1}{2}}}{\sqrt{\pi}}\sum_{m=0}^{\infty}\frac{a_m\,(2m)!!}{(2m+1)!!}x^m
\end{aligned}
\tag{7}
$$

where conventionally, $(-1)!!=1$. Given this general result, it is clear then that the half-integral of a function multiplies each term in the original series by a factor of $\frac{(2m)!!}{(2m+1)!!}$, while also scaling the whole series by a factor of $\frac{2x^{\frac{1}{2}}}{\sqrt{\pi}}$. Since Equation 2 holds, applying the half-integral operator to Equation 7 must yield:

$$
\left(J^{\frac{1}{2}}\left(J^{\frac{1}{2}}f\right)\right)(x)=\sum_{m=0}^{\infty}\frac{a_m}{m+1}x^{m+1}=\sum_{m=0}^{\infty}b_m x^m
\tag{8}
$$

where $b_0=0$. Having defined the series coefficient as $b_m=\frac{a_{m-1}}{m}$ for $m\geq 1$, we now have recovered a power series of the same analytic form as where we started. Thus, since we have already determined the impact that a half integral has on the power series of a proposed analytic function, and since we know that two half integrals leads to the whole integral in the conventional sense (which yields a different known power series), we can always determine the resultant series from taking a variable number of half-integrals.

The still quite general result given in Equation 7 can then be used to deduce the series solution to a variety of challenging integrals. For instance:

$$
\begin{aligned}
\int_0^1\frac{e^t}{\sqrt{1-t}} &= \sum_{m=0}^{\infty}\frac{2\,(2m)!!}{m!\,(2m+1)!!}=\sum_{m=0}^{\infty}\frac{2}{\left(\frac{3}{2}\right)_m} \\
\int_0^1\frac{\mathrm{Li}_s(t)}{\sqrt{1-t}} &= \sum_{m=1}^{\infty}\frac{2\,(2m)!!}{m^s\,(2m+1)!!}=\sum_{m=1}^{\infty}\frac{2m!}{m^s\left(\frac{3}{2}\right)_m} \\
\int_0^1\frac{\arcsin(t)}{\sqrt{1-t}} &= \sum_{m=0}^{\infty}\frac{2\,(m!)^2\left(\frac{1+(-1)^m}{2}\right)}{2^m\left(\left(\frac{m}{2}\right)!\right)^2\left(\frac{3}{2}\right)_{m+1}}
\end{aligned}
\tag{9}
$$

In general, for real α, the following power series represents the integral of order α:

$$\left(J^{\alpha}f\right)(x) = \frac{x^{\alpha}}{\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{a_m m!\, x^m}{(\alpha)_{m+1}} \tag{10}$$

The rich math then comes out in differentiating with respect α, which involves $\partial_{\alpha}\left(\frac{1}{(\alpha)_{m+1}}\right) = -\frac{\psi_0(m+\alpha+1)-\psi_0(\alpha)}{(\alpha)_{m+1}} = -\frac{\psi_0(m+\alpha+1)+\gamma}{(\alpha)_{m+1}}$ (where $\psi_n(t) = \partial_t^{n+1}\ln(\Gamma(t))$ is the nth order polygamma function, and γ is the Euler-Mascheroni constant), then setting $\alpha = 1$. Then, we find (p is an integer):

$$\begin{aligned}\left(\partial_{\alpha}^{p}\left(\Gamma(\alpha)\, J^{\alpha} f(t)\right)\right)_{\alpha\to 1} &= \int_0^x \mathrm{d}t\, \ln^p(x-t)\, f(t) \\ &= \sum_{k=0}^{p} \binom{p}{k} \ln^k(x) \int_0^x \mathrm{d}t\, \ln^{p-k}\left(1-\frac{t}{x}\right) f(t)\end{aligned} \tag{11}$$

Now let $p = 1$ and $x = 1$. Now we can do the following through integration by parts:

$$\begin{aligned}\left(\partial_{\alpha}\left(\Gamma(\alpha)\, J^{\alpha} f(t)\right)\right) &= \int_0^1 \mathrm{d}t\, \ln(1-t)\, f(t) \\ &= \left(-\mathrm{Li}_2(t)\, tf(t)\right)_{t\to 0}^{t\to 1} + \int_0^1 \mathrm{d}t\, \mathrm{Li}_2(t)\, \partial_t\left(tf(t)\right) \\ &= -\left(\sum_{m=0}^{\infty} \frac{a_m m!}{(\alpha)_{m+1}} \left(\psi_0(m+\alpha+1) - \psi_0(\alpha)\right)\right)_{\alpha\to 1} \\ &= -\sum_{m=0}^{\infty} \frac{a_m}{m+1}\left(\psi_0(m+2)+\gamma\right) \\ \implies \int_0^1 \mathrm{d}t\, \mathrm{Li}_2(t)\, \partial_t\left(tf(t)\right) &= \frac{\pi^2}{6} f(1) - \sum_{m=0}^{\infty} \frac{a_m}{m+1}\left(\psi_0(m+2)+\gamma\right)\end{aligned} \tag{12}$$

where it is assumed that $\lim_{t\to 0} \mathrm{Li}_2(t)\, tf(t) = 0$ such that $f(t)$ at worst goes $\sim \frac{1}{t}$. But of course $f(t)$ must be expressible as a positive integer power series for the method to work, so this case is already satisfied. Equation 12 is interesting, for one, because $\partial_t(tf(t))$ can be a general function which has an anti-derivative expressible as a power series about 0. Though the series is a conglomeration of complicated factors, the series solution is in itself a compact representation of the dilogarithm integral. This formula directly shows that for $f(t) = 1$, we get that since $\psi_0(2) = 1-\gamma$, the dilogarithm integral becomes:

$$\int_0^1 \mathrm{d}t\, \mathrm{Li}_2(t) = \frac{\pi^2}{6} - 1$$

as can confirmed trivially by direct methods, since $\partial_t \mathrm{Li}_2(t) = -\frac{\ln(1-t)}{t}$. However, this method is quite powerful, as can be exhibited if we integrate by parts repeatedly like:

$$\begin{aligned}\int_0^1 \mathrm{d}t\, \mathrm{Li}_n(t) \left[\left(\prod_{j=1}^{n-2} \partial_t\, t\right) \partial_t\left(tf(t)\right)\right] &= (-1)^{n-1} \sum_{q=1}^{n-1} (-1)^q \zeta(q+1) \left[\left(\prod_{j=1}^{q-1} t\, \partial_t\right) tf(t)\right]_{t\to 1} \\ &\quad + (-1)^{n-1}\left(\gamma \int_0^1 \mathrm{d}t\, f(t) + \sum_{m=0}^{\infty} \frac{a_m \psi_0(m+2)}{m+1}\right)\end{aligned} \tag{13}$$

where in the first line it is intended that the derivatives be evaluated on all functions of t in the square brackets from right to left. This formula covers a diverse range of functions, and the only remaining complexity is to choose what the term in square brackets comes to, then do $n-1$ anti-derivatives as in the product formula to find the corresponding $f(t)$ needed to determine a_m. Then, we have a sophisticated infinite series and a finite series corresponding to a compact, closed form for the integral of the product of the polylogarithm of positive integer order and a function of choice. For example, :

$$\int_0^1 \mathrm{d}t \, \mathrm{Li}_2(t) \cos(t) = \frac{\pi^2}{6} \sin(1) - \gamma \, \mathrm{Si}(1) - \sum_{m=0}^{\infty} \frac{(-1)^{\frac{m}{2}} \psi_0(m+2)}{m! \, (m+1)^2} \left(\frac{1+(-1)^m}{2} \right)$$

which is a one line solution to an integral with substantial complexity. Si is the sine integral as it is typically defined - $\mathrm{Si}(\infty) = \frac{\pi}{2}$. However, Equation 13 is incomplete - for the following converging integral, some interesting drawbacks are introduced:

$$\int_0^1 \mathrm{d}t \, \frac{\mathrm{Li}_2(t) \cos(t)}{t}$$

For this integral, $f(t) = \frac{\mathrm{Ci}(t)}{t}$, where $\mathrm{Ci}(t)$ is the cosine integral. The series for $f(t)$ is thus:

$$f(t) = \frac{\gamma}{t} + \frac{\ln(t)}{t} + \sum_{k=1}^{\infty} \frac{(-1)^k \, t^{2k-1}}{(2k)\,(2k)!}$$

which leads to the following representation:

$$\int_0^1 \mathrm{d}t \, \frac{\mathrm{Li}_2(t) \cos(t)}{t} = \frac{\pi^2}{6} \cos(1) - \sum_{m=0}^{\infty} \frac{a_m}{m+1} \left(\psi_0(m+2) + \gamma \right) \tag{14}$$

where suddenly we realize that the $f(t)$ cannot be easily represented by a series expansion about $t=0$. Before we get too connected to the idea that the right side expression converges (just as the integral does), we may want to return to the beginning of the calculation: refer to Equation 12. In this equation, substituting the $f(t)$ above gives a clearly diverging integral. So the original integral diverged before integration by parts, so any results devised from this operations are likely not valid. Furthermore, if $f(t)$ is expressible as a power series, then $\lim_{t\to 0} \mathrm{Li}_2(t) \, t f(t) = 0$ by definition. So, the inability to solve for a_m coefficient after calculation of $f(t)$ is a sure sign that this compact series solution method highlighted above cannot be used to solve the given integral. Now, I have two remaining ideas I'd like to explore before concluding:

- exploring the idea of having x remain undetermined in the α derivative calculation. This provides a particularly interesting case, as we will explore next.
- exploring if we can take two derivatives with respect to α and see the results.

To address the first point, reconsider Equation 12:

$$\frac{\Gamma(\alpha)}{x^\alpha} (J^\alpha f)(x) = \int_0^1 \mathrm{d}t \, (1-t)^{\alpha-1} f(xt) = \sum_{m=0}^{\infty} \frac{a_m \, m!}{(\alpha)_{m+1}} x^m$$

Now, we can get:

$$
\begin{aligned}
\left(\partial_\alpha\left(\frac{\Gamma(\alpha)}{x^\alpha}J^\alpha f(t)\right)\right)_{\alpha\to 1} &= \int_0^1 \mathrm{d}t\, \ln(1-t)\, f(xt) \\
&= -\gamma\int_0^1 \mathrm{d}t\, f(xt) - \sum_{m=0}^{\infty}\frac{a_m\, x^m}{m+1}\psi_0(m+2) \\
\implies \int_0^1 \mathrm{d}t\, \mathrm{Li}_2(t)\, \partial_t(t\, f(xt)) &= \frac{\pi^2}{6}f(x) - \gamma\int_0^1 \mathrm{d}t\, f(xt) - \sum_{m=0}^{\infty}\frac{a_m\, x^m}{m+1}\psi_0(m+2) \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 \mathrm{d}t\, \mathrm{Li}_n(t)\left[\left(\prod_{j=1}^{n-2}\partial_t\, t\right)\partial_t(tf(xt))\right] &= \sum_{q=1}^{n-1}(-1)^{n+q-1}\zeta(q+1)\left[\left(\prod_{j=1}^{q-1}t\,\partial_t\right)tf(xt)\right]_{t\to 1} \\
&+ (-1)^{n-1}\left(\gamma\int_0^1 \mathrm{d}t\, f(xt) + \sum_{m=0}^{\infty}\frac{a_m\, x^m}{m+1}\psi_0(m+2)\right)
\end{aligned}
$$

One thing to note is that this method will not help with solving integrals like the following:

$$
\int_0^x \mathrm{d}t\, \mathrm{Li}_2(t)\, \mathrm{Li}_2(1-t)
$$

Because the form of the integral will always take the following form:

$$
\int_0^x \mathrm{d}t\, \mathrm{Li}_2\left(\frac{t}{x}\right)\mathrm{Li}_2(1-t)
$$

which is inherently different. However, an interesting aspect of Equation 15 is that one can take derivatives with respect to x and get a number of integrals "for free". To see this idea in practice, consider $n=2$, $\partial_t(tf(xt)) = \partial_{xt}(xtf(xt)) = \frac{\mathrm{Li}_2(xt)}{xt}$ and $f(xt) = \frac{\mathrm{Li}_3(xt)}{xt}$. Then:

$$
I_1(x) = \int_0^1 \mathrm{d}t\, \frac{\mathrm{Li}_2(t)\,\mathrm{Li}_2(xt)}{t} = \frac{\pi^2}{6}\mathrm{Li}_3(x) - \gamma\mathrm{Li}_4(x) - \sum_{m=0}^{\infty}\frac{\psi_0(m+2)}{(m+1)^4}x^{m+1} \qquad (16)
$$

Then, we have the following integrals easily by differentiation with respect to x for all positive integers p:

$$
\int_0^1 \mathrm{d}t\, \frac{\mathrm{Li}_2(t)\,\mathrm{Li}_{2-p}(xt)}{t} = \frac{\pi^2}{6}\mathrm{Li}_{3-p}(x) - \gamma\mathrm{Li}_{4-p}(x) - \sum_{m=0}^{\infty}\frac{\psi_0(m+2)}{(m+1)^{4-p}}x^{m+1} \qquad (17)
$$

This formula indeed demonstrates the power of this method for $\mid x \mid < 1$. To address the second idea in the point list above, we set $x=1$ and differentiate with respect to α more than once, then set $\alpha = 1$. This gives:

$$
\int_0^1 \mathrm{d}t\, \ln^2(1-t)\, f(t) = \sum_{m=0}^{\infty}\frac{a_m}{m+1}\left((\psi_0(m+2)+\gamma)^2 - \left(\psi_1(m+2) - \frac{\pi^2}{6}\right)\right) \qquad (18)
$$

where $\psi_1(1) = \zeta(2) = \frac{\pi^2}{6}$. Another example is:

$$
\begin{aligned}
\int_0^1 \mathrm{d}t\, \ln^3(1-t)\, f(t) = &-\sum_{m=0}^{\infty}\frac{a_m}{m+1}\left((\psi_0(m+2)+\gamma)^3 + \psi_2(m+2) - \psi_2(1)\right) \\
&+ 3\sum_{m=0}^{\infty}\frac{a_m}{m+1}\left((\psi_0(m+2)+\gamma)\left(\psi_1(m+2) - \frac{\pi^2}{6}\right)\right)
\end{aligned} \qquad (19)
$$

This concludes this blog post.