Title: Decoupling Fourier Coefficient Series
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Recall the Fourier Coefficient definition we've been using lately:

$$c_n = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} g\left(x'\right) e^{-2\pi i n \frac{x'}{L}} \mathrm{d}x' \tag{1}$$

and the resulting Fourier Series is defined by:

$$g\left(x\right) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n \frac{x}{L}} \tag{2}$$

Through expanding Equation 1 as a Taylor Series, we obtained the following in the last post:

$$\begin{aligned} c_n &= -\sum_{m=0}^{\infty}\sum_{q=0}^{m}\left(\frac{L}{2}\right)^m \left(\frac{1}{i\pi n}\right)^{q+1} \frac{g^{(m)}\left(0\right)}{(m-q)!}\frac{\left(1-(-1)^{m-q}\right)}{2}\left(-1\right)^n \\ n\frac{\partial c_n}{\partial n} &= \sum_{m=0}^{\infty}\sum_{q=0}^{m}\left(\frac{L}{2}\right)^m \left(\frac{1}{i\pi n}\right)^{q} \frac{g^{(m)}\left(0\right)}{(m-q+1)!}\left(m+1\right)\frac{\left(1-(-1)^{m-q+1}\right)}{2}\left(-1\right)^n \\ L\frac{\partial c_n}{\partial L} &= -\sum_{m=0}^{\infty}\sum_{q=0}^{m}\left(\frac{L}{2}\right)^{m+1} \left(\frac{1}{i\pi n}\right)^{q+1} \frac{g^{(m+1)}\left(0\right)}{(m-q+1)!}\left(m+1\right)\frac{\left(1-(-1)^{m-q+1}\right)}{2}\left(-1\right)^n \end{aligned} \tag{3}$$

In these cases, the second sum is coupled to the first sum in the form of the upper bound. To decouple these sums, we switch the order of summation and then index the sum in m down by q. Then, we attain:

$$\begin{aligned} c_n &= -\sum_{m=0}^{\infty}\sum_{q=0}^{\infty}\left(\frac{L}{2}\right)^{m+q} \left(\frac{1}{i\pi n}\right)^{q+1} \frac{g^{(m+q)}\left(0\right)}{m!}\frac{\left(1-(-1)^{m}\right)}{2}\left(-1\right)^n \\ n\frac{\partial c_n}{\partial n} &= \sum_{m=0}^{\infty}\sum_{q=0}^{\infty}\left(\frac{L}{2}\right)^{m+q} \left(\frac{1}{i\pi n}\right)^{q} \frac{g^{(m+q)}\left(0\right)}{(m+1)!}\left(m+q+1\right)\frac{\left(1-(-1)^{m+1}\right)}{2}\left(-1\right)^n \\ L\frac{\partial c_n}{\partial L} &= -\sum_{m=0}^{\infty}\sum_{q=0}^{\infty}\left(\frac{L}{2}\right)^{m+q+1} \left(\frac{1}{i\pi n}\right)^{q+1} \frac{g^{(m+q+1)}\left(0\right)}{(m+1)!}\left(m+q+1\right)\frac{\left(1-(-1)^{m+1}\right)}{2}\left(-1\right)^n \end{aligned} \tag{4}$$

To verify that these formulas are correct, recall the partial differential equation satisfied by these terms:

$$\left(-1\right)^n \frac{g\left(\frac{L}{2}\right)+g\left(-\frac{L}{2}\right)}{2} = c_n + n\frac{\partial c_n}{\partial n} + L\frac{\partial c_n}{\partial L} \tag{5}$$

Dividing both sides by $(-1)^n$, we get for the left side:

$$\frac{g\left(\frac{L}{2}\right)+g\left(-\frac{L}{2}\right)}{2}=\sum_{m=0}^{\infty}\frac{g^{(m)}(0)}{m!}\left(\frac{L}{2}\right)^m\left(\frac{1+(-1)^m}{2}\right)=\sum_{m=0}^{\infty}\frac{g^{(2m)}(0)}{(2m)!}\left(\frac{L}{2}\right)^{2m} \tag{6}$$

From Equation 4, it is easy to show that:

$$(-1)^n n\frac{\partial c_n}{\partial n}=\sum_{m=0}^{\infty}\left(\frac{L}{2}\right)^{2m}\frac{g^{(2m)}(0)}{(2m)!}+\sum_{m=0}^{\infty}\sum_{q=0}^{\infty}\left(\frac{L}{2}\right)^{2m+q+1}\frac{g^{(2m+q+1)}(0)}{(i\pi n)^{q+1}}\frac{(2m+q+2)}{(2m+1)!} \tag{7}$$

It is then straightforward to show that only the first term on the right hand side of Equation 7 survives after substituting into Equation 5. This serves as one verification that Equation 4 is indeed the correct - substituting various known c_n and $g(x)$ pairs is another way to verify Equation 4's correctness - I leave this to the reader.

Having verified the identity in question, we will now use it to solve for c_n associated with a given set of $g^{(m)}(0)$ coefficients. We will then use the same coefficients to solve for $g(x)$ for the left side of Equation 2. Then, substituting $g(x)$ and c_n into Equation 2, we may arrive at interesting series identities for n. Consider first the example of $g^{(m)}(0)=(k)_m$, where $0.5<k<1$ is a real number and $(k)_m$ is the falling factorial of k of order m. Noting first that $g^{(m+q)}(0)=(k)_{m+q}=(k)_m(k-m)_q$, we solve for c_n and $g(x)$:

$$\begin{aligned}
c_n&=-\sum_{m=0}^{\infty}\frac{\left(\frac{L}{2}\right)^{2m+1}(k)_{2m+1}}{(2m+1)!}\frac{(-1)^n}{i\pi n}\sum_{q=0}^{\infty}(k-2m-1)_q\left(\frac{L}{2\pi in}\right)^q\\
&=-\sum_{m=0}^{\infty}\frac{\left(\frac{L}{2}\right)^{2m+1}(k)_{2m+1}}{(2m+1)!}\frac{(-1)^n}{i\pi n}F_{20}\left(1,2m+1-k;;-\frac{L}{2\pi in}\right)\ \text{for}\ \mid n\mid\geq 1\\
g(x)&=F_{10}(-k;;-x)
\end{aligned} \tag{8}$$

where $c_0=\frac{1}{2}\left(F_{21}\left(1,-k;2;-\frac{L}{2}\right)+F_{21}\left(1,-k;2;\frac{L}{2}\right)\right)$. The outcome of this example then involves combining these results into Equation 2, then setting $x=0$ and $L=1$:

$$\begin{aligned}
1=&\frac{1}{2}\left(F_{21}\left(1,-k;2;-\frac{1}{2}\right)+F_{21}\left(1,-k;2;\frac{1}{2}\right)\right)\\
&-\sum_{n=-\infty}^{\infty}\sum_{m=0}^{\infty}\frac{\left(\frac{1}{2}\right)^{2m+1}(k)_{2m+1}}{(2m+1)!}\frac{(-1)^n}{i\pi n}F_{20}\left(1,2m+1-k;;-\frac{1}{2\pi in}\right)
\end{aligned} \tag{9}$$

I have verified Equation 9 for $k=1$ and $k=2$, as well as numerically for irrational numbers such as $k=\frac{1}{\pi}$. This is comforting numerical confirmation. I will conclude with two comments on what has been done thus far:

- the utility of Equation 4 is that you can consider $g^{(m)}(0)$ that lead to a given c_n for which you wish to find the Fourier Series solution. However, some early efforts have been futile for finding some solutions c_n as they appear to not all fit well with the form of c_n in Equation 4.

- it is indeed true (and *noteworthy*) that not all series given by c_n will correspond to a Fourier Series function $g(x)$. In a function space known as L_2, the model Fourier Series must be *square summable*. For example, $c_n=\frac{(-1)^{n-1}}{\sqrt{n}}$ is not square summable since $\sum_{n=-\infty}^{\infty}c_n^2$ diverges. This is a result of Parseval's Theorem in a function space.

This concludes this blog post.